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FIRST COURSE IN ALGEBRA

BY

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PREFACE

ALGEBRA is essentially a body of principles by the aid of which we can discover the information implicitly given in certain kinds of statements. These statements are for the most part descriptions of numbers and can be expressed in the form of equations. It is the function of a text-book in elementary algebra to show the student how he can use these principles in identifying numbers thus described.

The present book is the result of an attempt to embody this conception of algebra in a form suitable for first-year high school pupils. Everything has accordingly been made subordinate to the equation and the solution of problems.

In carrying out this general plan an effort has been made to present the subject in as informal a way as possible. For example, the usual appeal to axioms to justify certain operations with equations has been omitted and the argument has been based on the definitions of the fundamental operations. Of course this involves assumptions which are equivalent to the axioms, but the assumptions are so plausible that it seems best not to give a formal statement of them in a book for beginners.

Many new problems will be found in the book, together with some old ones in new forms. The informational problems serve to relate the subject to geometry, physics, and everyday life.

The notion of the abscissa of a point is an extremely simple one, and effective use can be made of it to bring out the close connection between algebra and geometry.

PREFACE

The use of graphical methods has been explained as ful
seemed advisable in a book for beginners. Nothing has
said of the loci of equations of higher than the first de
Some of these will be explained in the Second Course.

The order of topics has been determined on the obv
principle that the simpler ones should come first. Thus
reference is made to negative numbers until Chapter III, a
the student has acquired a certain degree of familiarity w
the algebraic notation. The history of the slow introduct
of these numbers into mathematics indicates the inherent di
culty of the subject and warns us against putting it before t
student while he is adjusting himself to the new and stran
literal notation. Such subjects as fractions and division
polynomials are placed later than usual because of their form
difficulties.

The discussion of imaginary numbers has been left to th
Second Course. These numbers cannot be treated adequatel
in a beginner's book, and it seemed better therefore to omi
them altogether.

Some problems which are too difficult for the average stu
dent have been inserted in order to afford opportunity to the
brighter students to exercise their powers.

There is sufficient material in the body of the book for a full
year's course. Certain special topics which are required by
the Regents of the State of New York are given in the appendix. They will also be given in the Second Course.

The diagram on page 111 is a record of temperatures
at the meteorological observatory of the United States
Bureau in Central Park, New York City, and
through the kindness of Mr. James H. Scarr
caster.

A number of high school teacher
various parts of the country have
fully and critically, and I have
tions in making the final

especially valuable in enabling me the better to adapt the book to the needs of first-year pupils.

Professors W. B. Carver of Cornell University, H. E. Cobb of Lewis Institute, Chicago, and P. R. Dean of the Curtis High School, New York City, read the manuscript and made many suggestions which have been of great value to me.

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FIRST COURSE IN ALGEBRA

CHAPTER I

INTRODUCTION

1. It frequently happens that reflection enables us to get more information from a statement than appears on the surface. Thus if we are told that A is 20 years old and that 2 years ago he was twice as old as his brother, we can find out how old the brother is. From the statement that the distance around a rectangular field containing 5 acres is 120 rods, we can, with some effort, discover how long the field is. (See p. 165.) If we had some electric lights and a motor attached to the same meter, and if we knew that the bill for one month (30 days), when the motor had been running every day and the lights had been used 10 days, was \$1.12, and \$2.52 another month (30 days), when the lights and the motor had been in use every day, we would have sufficient information to determine the cost per month of running the motor. It would be easy to cite statements containing information still more hidden. We shall meet many such later in this book.

2. Usefulness of Algebra. — Since much important information comes to us only in this indirect way, it is desirable for us to have a systematic means for discovering what is really contained in a statement; and we shall find in **algebra** such a systematic means for bringing to light certain kinds of information. How algebra can be used for this purpose will be illustrated in a simple way by the problems at the end of this chapter.

3. Numbers represented by Letters. — In solving these problems the student will find it desirable to represent certain numbers by letters, and he will see as he proceeds that such a representation of numbers is common in algebra. The use of letters makes it easier for us to reason about complicated matters and to express our results in a simple and concise way.

4. Addition. — The sign + placed between two numbers indicates, as in arithmetic, that the numbers are to be added.

Thus, $a + b$ indicates the sum of the numbers represented by a and b.

EXERCISES

1. Represent the sum of of m and n; that is, the sum of the numbers represented by m and n.

2. Represent the sum of a and x; a and 2; m and 5.

3. What is the value of $a + 2$ when a is 10; that is, what is the number represented by $a + 2$ when a represents 10?

4. What is the value of $m + 5$ when m is 7?

5. If $x = 3$, $x + 15 = ?$ $x + 24 = ?$ $x + 81 = ?$

(The symbol = placed between two numbers indicates that they are equal. Thus, $x = 3$ is read "x equals 3." The first question of Ex. 5 can be read "If x equals 3, what is the value of $x + 15$?")

6. If A is 20 years old, how old will he be in x years? in y years? in n years?

7. If A is x years old, how old will he be in 10 years? in a years? in b years?

8. What will be the answers to Ex. 7 if $x = 30$? $x = 5$? $x = 15$?

9. What is the perimeter of a triangle the lengths of whose sides are 8 inches, 10 inches, and 12 inches respectively?

The perimeter of a figure is the sum of the lengths of its sides.

10. The distance called for in Ex. 9 is equal to $8+10+12$. Can you make use of this fact to express the perimeter of a triangle the lengths of whose sides are x, y, and z respectively?

11. What is the perimeter of a triangle the lengths of whose sides are a, b, and c?

12. What is the perimeter of a triangle the lengths of whose sides are 8, 4, and x respectively?

13. The length of one side of a triangle is x inches. The second side is 2 inches longer than this, and the third side is 7 inches longer than the second side. What are the lengths of these two sides?

14. What is the perimeter of the triangle of Ex. 12 if $x = 10$?

15. A rectangle is 2 feet longer than it is wide. What is its length if it is x feet wide?

16. What is the perimeter of a rectangle a feet long and b feet wide?

17. St. Louis is 263 miles farther from Cleveland than Indianapolis is. If d is the distance between Cleveland and Indianapolis, what is the distance between Cleveland and St. Louis?

5. Subtraction. — The sign $-$ placed between two numbers indicates, as in arithmetic, that the second number is to be subtracted from the first one.

Thus, $a-b$ indicates that the number represented by b is to be subtracted from the number represented by a; or, as we may say more briefly, that b is to be subtracted from a.

EXERCISES

1. Indicate that 2 is to be subtracted from x.

2. What is the value of this difference when x stands for 17?

3. Indicate that x is to be subtracted from m.

4 Indicate that m is to be subtracted from x.

5. Represent the number that is 9 less than y.

6. What is this number when $y = 28\frac{1}{2}$?

7. What is the smaller of two numbers if the larger one is x and the difference is 6?

8. If the sum of two numbers is 29 and one of them is 8, what is the other one?

9. If the sum of two numbers is s and one of them is 8, what is the other one?

10. If the sum of two numbers is s and one of them is a, what is the other one?

11. The three sides of a triangle are 5 inches, 9 inches, and 12 inches respectively. By how much does the sum of the first two sides exceed the third side?

This excess is evidently $5 + 9 - 12$.

12. If the three sides of a triangle are a, b, and c respectively, by how much does the sum of the first two sides exceed the third side?

13. The sum of two numbers is 34 and the smaller one is 7. What is the larger one and what is their difference?

14. The sum of two numbers is s and the smaller one is 5. What is the larger one and what is their difference?

15. The sum of two numbers is 12 and the smaller one is b. What is the larger one and what is their difference?

16. The sum of two numbers is 36 and the smaller one is a. What is the larger one and what is their difference?

17. If the sum of two numbers is s and the smaller one is a, what is their difference?

6. **Multiplication.** — It will be recalled that in arithmetic the number resulting from multiplying two or more numbers together is called the **product** of these numbers, and that the

numbers multiplied together to form the product are called **factors** of the product.

For example, the product of 4, 5, and 7 is 140; that of 9 and x is $9x$; and that of 2, a, b, and c is $2abc$.

Any factor of a product that is represented by a letter or letters is called a **literal** factor. When numbers occur as factors they are called **numerical** factors.

Thus, in the last example, 2 is a numerical factor and a, b, and c are literal factors.

7. Coefficients. — The factors may be taken in any order and grouped in any way, and the resulting product will always be the same.

Thus, $3 \times 5 \times 7 = 5 \times 3 \times 7 = 7 \times 3 \times 5 = 15 \times 7 = 3 \times 35$; and $2abc = 2bac = c2ba = 2c \times ba$.

Any product can be looked upon as the product of two factors, and either of these factors is called the **coefficient** of the other.

In the product $2abc$, for example, we can say that $2c$ is the coefficient of ab.

It is usual to write the numerical factor first, if one occurs, and the sign of multiplication is omitted between two literal factors and between a literal factor and a numerical factor. We agree that abc shall be the same as $1abc$, and in general that any product which seems to have no numerical coefficient shall be considered to have the coefficient 1.

1. If n represents a number, what will represent 7 times this number? 12 times this number? 3 times this number?

2. Find the value of each of these numbers when $n = 4$; 8; 21.

3. If you walk 4 miles an hour, how far will you go in 3 hours? in x hours? in h hours?

4. If n is the number of miles from New Orleans to Chicago, what is the distance from New Orleans to the point midway between these places?

5. If a man's salary is d dollars a month, how much does he receive a year? How much is this when d is 85? 120? 170?

6. Describe in words what $4x$, $10x$, $2n$, and $17a$ represent.

7. If $x = 7$, what does $4x$ equal? $10x$?

8. If $n = \frac{5}{2}$, $2n = ?$ If $a = 3$, $17a = ?$

9. A bushel of shelled corn weighs 56 pounds. How much will b bushels weigh? $2n$ bushels? $5x$ bushels?

10. If one of two numbers is represented by x and the other one is 6 times as large, how can you represent their sum?

Since the two numbers are x and $6x$, their sum is $x + 6x$, or $7x$.

11. If one of two numbers is represented by n and the other one is 4 times as large, how can you represent their sum?

12. If one of two numbers is represented by x and the other one is 5 times as large, how can you represent their difference?

13. If one of two numbers is 3 times the other, which is represented by x, how can you represent their sum? their difference?

14. How many inches are there in f feet? in n feet? in $3x$ feet?

15. How many ounces are there in x pounds, y ounces?

16. By how much do a yards exceed 2 feet?

17. One box weighs p pounds and another one a ounces. What is the difference in their weights, the first one being the heavier one?

18. The steamship *Mauretania* holds the record for the fastest day's run (24 hours). This is 676 knots, and was made in January, 1911. The battleship speed record is held by the

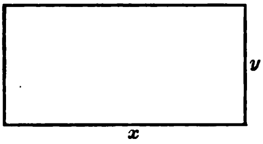
Wyoming. If this is k knots per hour, by how much would a day's record run of the *Wyoming* fall short of that of the *Mauretania ?*

19. What is this result if $k = 22.045$?

Area of a Rectangle. — If a rectangle is 9 inches long and 3 inches wide, its area is 9×3 (or 27) square inches, and in general the number of square units in a rectangle is equal to the product of its length by its width.

If A denotes the area, x the length, and y the width, of a rectangle, we can make use of the literal notation of algebra to express this more concisely thus:

$$A = xy.$$



20. What is the area of a rectangle a inches long and 8 inches wide?

21. A rectangle is a feet by b feet. How many square inches does it contain?

22. One rectangle is m feet by n feet and another one is x feet by y feet. What is the sum of their areas?

23. What is the sum of the areas in square inches of the rectangles of Ex. 22 if the second one is x inches by y inches?

24. How many square rods are contained in 3 rectangular lots each a rods by b rods and 4 lots each 5 rods by m rods?

25. How many more square rods are contained in the first 3 lots of Ex. 24 than in the last 4?

Volume of a Box. — If a box is 9 inches long, 3 inches wide, and 4 inches high, it contains $9 \times 3 \times 4$ (or 108) cubic inches, and in general the number of cubic units in a box is equal to the product of its length, width, and height.

If V denotes the volume, x the length, y the width, and z the height, we can state this fact thus:

$$V = xyz.$$

26. What is the volume of a box that is 2 by 3 by x?

27. How many cubic feet of earth would it take to raise the level of the first 3 lots of Ex. 24 by 1 foot?

28. How many cubic inches are there in two boxes which are a feet by b feet by c feet and x inches by y inches by z inches respectively?

29. How much larger is the first box of Ex. 28 than the second one?

30. What is the volume in cubic inches of a box y yards long, f feet wide, and 32 inches high?

It is worth while to note here an important difference between the literal notation of algebra and the Arabic notation used in arithmetic. In the former, as has just been explained, we indicate the product of two expressions by writing one after the other with no sign of operation between, whereas, in the latter the sum of two numbers is often indicated in this way.

Thus, ab represents the product of a and b, but $4\frac{3}{7}$ represents the sum of 4 and $\frac{3}{7}$, and 25 represents the sum of 20 and 5 (the 2 stands for 20 here because it is in the tens' place).

HISTORICAL NOTE. — The Hindu Aryabhatta, born 476 A. D., indicated that one number was to be added to another by writing the numbers in succession with no sign between them.

8. Exponents. — It frequently happens that several factors of a product are the same. In such a case the repeated factor is written only once with a number above and to the right of it to indicate the number of times it occurs as a factor.

Thus, $4\,aaxxx$ is written $4\,a^2x^3$ and is read "four a squared x cubed."

The number thus placed above and to the right of a factor is called an **exponent**. It indicates the number of times a factor occurs in a given product. If no exponent is written after a factor, the exponent 1 is understood.

1. What is the area of a square each side of which is x inches?

2. One side of a square is a inches. What is the area of a square 4 times as large?

3. The side of one square is x inches and the side of another one is 5 inches. What is the sum of their areas?

4. How many more square inches are contained in 2 squares like the first one of Ex. 3 than in the second one?

5. What is the sum of the areas of two squares whose sides are a and b respectively?

6. How much larger is the first square of Ex. 5 than the second one?

7. The sides of 3 squares are a yards, b feet, and c inches respectively. What is the sum of their areas in square inches?

8. What is the area of a square that is 6 square inches larger than the sum of 2 squares, each of whose sides is x inches?

9. What is the area of this square if $x = 7$?

10. What is the volume of a cube whose edge is 4 inches?

11. What is the sum of the volumes of 2 cubes whose edges are a and 5 respectively?

12. How many cubic feet are there in a box h feet by h feet by x feet?

13. How many square feet in the surface of the box of Ex. 12?

9. Equations. — Many of the statements that we shall have occasion to make will be to the effect that one combination of numbers is equal to another one.

Statements of this kind are called **equations.**

Thus, $$3x - 4 + 2x = 6 - 5x$$
is an equation.

10. Solution of Problems. — We are now in a position to illustrate how algebra can be used in a simple way to gain information.

From the information that there are 531 votes in the Electoral College we wish to find out how many votes a candidate must have in order to defeat his opponent by 55, it being assumed that there are only two candidates.

Since the successful candidate is to have 55 votes more than his opponent, the total number of votes must be 55 less than twice the number cast for the successful candidate. Hence twice this number must be 586. The successful candidate must therefore get half this many, or 293 votes.

This is the desired information. But if we had represented the number of votes necessary for the successful candidate by some letter, as n, we could have expressed our reasoning more concisely.

For then $n - 55 =$ number of votes for the unsuccessful candidate,
and $n - 55 + n =$ the total number of votes cast for the two candidates.

But there were 531 votes altogether.

Hence,
$$n - 55 + n = 531,$$
or
$$2n - 55 = 531,$$
$$2n = 586,$$
$$n = 293.$$

PROBLEMS

1. The greater of two numbers exceeds the smaller by 5 and their sum is 75. What are the numbers?

2. The sum of two numbers is 60 and one is 12 less than the other one. What are the numbers?

3. One number is 3 times the other one and their sum is 32. What are the numbers?

4. One number exceeds 4 times another one by 6, and their difference is 21. What are the numbers?

5. A rectangle is 3 times as long as it is wide and its perimeter is 40 inches. What are the dimensions of the rectangle?

6. A square and an equilateral triangle (that is, one with equal sides) are drawn on the same line, and the perimeter of the square exceeds that of the triangle by 2 inches. How long is a side of each of the figures?

7. The width of a rectangular field is 15 rods less than the length and the perimeter is 150 rods. What is the length?

8. Three fifths of the area of the original thirteen states was 894,405 square miles (approximately). What was the total area?

Hint. If $x =$ the total number of square miles,

then $\frac{3}{5}x = 894{,}405.$

Hence, $x = 894{,}405 \div \frac{3}{5}.$

9. Three fourths of the total area of the United States is 2,807,508 square miles. What is the total area?

10. Find two consecutive integers whose sum is 33.

11. A body of 100 students is to be divided into two groups, and the second group is to contain 30 more than the first one. How many will there be in the first group?

12. One side of a triangle is 3 inches longer than each of the other sides, and the perimeter is 29 inches. Find the lengths of the sides.

13. A man has two houses for which he wishes an annual rental of $1080. He thinks one of them is worth $10 per month more than the other. How much should he ask for each of them?

14. How much rent should he ask for each house if the first one is worth twice as much as the other one?

15. In a newspaper report of an election for mayor of a city, it was stated that B won over A by 117 votes in a total vote of 4629. What was the vote of each?

16. A lady has $800 with which to refurnish her living room and dining room. She decides to put $200 more into the former than into the latter. How much can she spend on each?

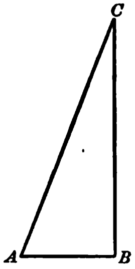
17. The combined cost of an upper and a lower berth in a sleeping car from Buffalo to Dayton, Ohio, is $4.05, and the upper berth costs 45 cents less than the lower. What is the cost of each?

18. In a combination of platinum and silver weighing 10 ounces, the platinum weighs 7 ounces more than the silver. What is the weight of each?

19. A, B, and C enter into a partnership requiring $20,000 capital. If A is to put in three times as much as B, and C as much as A and B together, what is the investment of each?

The Angles of a Triangle. — It is proved in geometry that the sum of the angles of a triangle is 180°.

20. In a triangle *ABC*, the angle *B* is 90°, and *A* is four times *C*. How many degrees are there in the angle *C*?



21. In a triangle *ABC*, the angle *B* is twice as large as the angle *A* and the angle *C* is three times as large as *A*. How many degrees are there in each?

22. In a triangle *ABC* the angle *A* is 90° and the angle *B* exceeds *C* by 18°. How many degrees are there in each of the angles *B* and *C*?

23. In a triangle *ABC* the angles *B* and *C* are equal and each exceeds *A* by 30°. How many degrees are there in each angle?

11. Many of the statements from which we need to gain further information are more complicated than these, and in order to be able to discover the information contained in them it will be necessary for the student to learn something of the methods of algebra. These methods are explained in the following chapters.

CHAPTER II

ADDITION AND SUBTRACTION

ADDITION

12. Monomials. — A **monomial** is a number or the product of numbers and literal factors.

Thus, 140, $9x$, and abc are all monomials. Such expressions as $a(b + c)$ represent the product of the number a and the sum of the numbers b and c. They are therefore, in a sense, the product of two literal factors. But we do not usually speak of them as monomials, although there are times when it is convenient to do so. (See Exercises 33–35 of the next article.)

13. If we add three times a number to seven times the same number the result is evidently ten times this number. The student will recognize that this is true no matter what number we take. Now when we wish to make a statement such as this one, that applies to all numbers, and also when we wish to make a statement about numbers whose value we do not care to give, it is customary to represent these numbers by letters.

Thus, if we use x to represent the number referred to in the preceding statement, this statement becomes $3x + 7x = 10x$. If we should use some other letter as m to represent this number, the statement becomes $3m + 7m = 10m$.

This illustration is a particular case of the following

Principle. — *The sum of several monomials that contain the same literal factors is obtained by multiplying the product of these factors by the sum of the coefficients of this product in the respective monomials. If there are no literal factors in these monomials, the sum is formed as in arithmetic.*

Example: $12\,xy + 5\,xy + 10\,xy = 27\,xy.$

The work may be arranged thus:

$$\begin{array}{rl} 12\,xy & \\ 5\,xy & \\ \underline{10\,xy} & \\ 27\,xy & \textit{Sum.} \end{array}$$

EXERCISES

Find the following indicated sums:

1. $14 + 5 + 7.$
2. $3\,a + a + 17\,a.$
3. $9\,b + 2\,b + 12\,b.$
4. $12\,xy + 7\,xy + 20\,xy + 19\,xy.$
5. $24\,m + 5\,m + 10\,m + 15\,m.$
6. $9\,x + 150\,x + 31\,x + 3\,x + x.$
7. $ac + ac + ac + ac + ac.$
8. $22\,y + y + 3\,y + 38\,y.$
9. $20\,abc + abc + 2\,abc + 12\,abc.$
10. $xyz + 50\,xyz + xyz.$
11. $12\,ab + 12\,ab + 12\,ab.$
12. $5\,a + 2\,a + 4\,a + 9\,a + 8\,a.$
13. $7\,abcxy + 3\,abcxy + 13\,abcxy.$
14. $21\,xyz + 6\,xyz + 2\,xyz.$
15. $y + y + y + y + y.$
16. $20\,a^2bc + a^2bc + 9\,a^2bc.$
17. $x^2y^2z^2 + x^2y^2z^2 + x^2y^2z^2 + x^2y^2z^2.$
18. $r^4s^3 + 5\,r^4s^3 + 16\,r^4s^3.$
19. $a + a + a + 3\,a.$
20. $a^5x^3y^2 + 4\,a^5x^3y^2 + 10\,a^5x^3y^2 + 15\,a^5x^3y^2.$
21. $21\,l^6m^5 + 2\,l^6m^5 + 10\,l^6m^5.$
22. $14\,a^3r^3 + 25\,a^3r^3.$
23. $126\,x + 4627\,x + 568\,x + 922\,x.$
24. $467\,a^3x^2 + 467\,a^3x^2 + 467\,a^3x^2.$
25. $1000\,b + 100\,b + 10\,b + b.$
26. $3\tfrac{1}{2}\,x^2y^3 + 4\tfrac{1}{2}\,x^2y^3 + 9\tfrac{3}{4}\,x^2y^3.$
27. $7\tfrac{2}{3}\,lmn + \tfrac{5}{6}\,lmn + lmn.$
28. $3.5\,b^3c^3 + 21.6\,b^3c^3 + 2\,b^3c^3 + 4.65\,b^3c^3.$

29. $1.27\,axy + 12.7\,axy + .127\,axy$.

30. $5\,a^xb^y + 27\,a^xb^y$.

31. $21\,l^xm^yn^z + 106\,l^xm^yn^z + 36\,l^xm^yn^z$.

32. $x^a + 5\,x^a + 100\,x^a + 29\,x^a + 137\,x^a$.

33. $3(b+c) + 2(b+c)$.

Consider $b + c$ as a literal factor. Then we have

$$3(b+c) + 2(b+c) = 5(b+c).$$

34. $10(3\,x+y) + 7(3\,x+y) + 16(3\,x+y)$.

35. $6(a+b+c) + 8(a+b+c) + 12(a+b+c)$.

14. Similar Monomials. — Two monomials are said to be **similar** if they contain the same literal factors.

Thus, $14\,a^3b^2c$ and $2\,a^3b^2c$ are similar, while $7\,x^4yz^2$ and $3\,x^3yz$ are dissimilar, as are $7\,x^4yz^2$ and $15\,abc^3$ and also $7\,x^4yz^2$ and 14. Two numbers are similar monomials.

The student has already learned how to add numbers represented by similar monomials. But he will have frequent occasion to consider the sum of numbers represented by dissimilar monomials.

15. Algebraic Expressions. — An **algebraic expression** is a combination of numbers, letters, and signs of operation such that its value can be found in accordance with the established rules of algebra when definite values are given to the letters.

16. Polynomials. — The indicated sum of two or more monomials is an algebraic expression called a **polynomial**.

The monomials are called the **terms** of the polynomial.

For example, $14\,a^3b^2c + 7\,x^4yz^2 + 8\,abc^3$ is a polynomial with the three terms $14\,a^3b^2c$, $7\,x^4yz^2$, and $8\,abc^3$.

If two terms of a polynomial are similar monomials, they are said to be **similar terms**.

Thus, $4\,a^2b$ and $7\,a^2b$ are similar terms in $a^3 + 4\,a^2b + ab^2 + 7\,a^2b + b^3$.

17. Binomials and Trinomials. — A polynomial of two terms is called a **binomial**; one of three terms is called a **trinomial**.

Thus, $4x^3y^2 + 7xy^4$ is a binomial and $7a + 3b + 9c$ is a trinomial. Sometimes the word polynomial is used in a more general sense to include monomials also.

18. Addition of Polynomials. — Since a polynomial is itself a sum of terms, the sum of two or more polynomials is a polynomial whose terms are the terms of the given polynomials. Thus, the sum of $3x + 4y + 8z$, $4x + 2y + 3z$, and $x + y + z$ is $3x + 4y + 8z + 4x + 2y + 3z + x + y + z$. But addition is such that the sum of two or more terms is the same whatever the order in which these terms are added, or, as is usually said, addition is **commutative**. Hence the similar monomials in the sum can be brought together and added in accordance with the rule for the addition of similar monomials. When this is done the sum given above becomes $3x + 4x + x + 4y + 2y + y + 8z + 3z + z$ or $8x + 7y + 12z$.

In order to get the sum in this final simple form as directly as possible it is advisable to apply the following

Rule. — *In addition write the polynomials in rows with similar monomials in the same columns and form the polynomial whose terms are the sums of the monomials in the several columns.*

This is illustrated by the example just given:

$$\begin{array}{lr} & 3x + 4y + 8z \\ & 4x + 2y + 3z \\ & x + y + z \\ \hline \textit{The sum is} & 8x + 7y + 12z \end{array}$$

19. Check for Addition. — The sum of polynomials should have the same numerical value as the sum of the numerical values of the polynomials when any numbers are taken as the values of the literal factors involved.

If we take 2, 3, and 5 as the values of x, y, and z respectively in the preceding illustration, we have

$$\begin{aligned} 3x + 4y + 8z &= 58, \\ 4x + 2y + 3z &= 29, \\ x + y + z &= 10. \end{aligned}$$

The sum of these numbers is 97 and also $8x + 7y + 12z = 97$.

This is as it should be, and the fact that these two results are the same serves as a check on our work in getting the sum of these polynomials. The student is warned, however, that this numerical test is only a check and is not infallible.

For example, suppose we were finding the sum of $2x + 3y + 6z$ and $x + 5y + z$, and that we arranged our work as follows:

Solution and check for $x = 3$, $y = 1$, $z = 2$.

$$\begin{array}{lll} & 2x + 3y + 6z = 6 + 3 + 12 = & 21 \\ & \underline{x + 5y + z} = 3 + 5 + 2 = & \underline{10} \\ \textit{Sum} & 3x + 10y + 6z & 31 \\ \text{But} & 3x + 10y + 6z = 9 + 10 + 12 = 31. & \end{array}$$

We might conclude, therefore, that we had found the correct sum, whereas, as a matter of fact, this sum is incorrect. However, a situation like this is comparatively rare and the chances are small that a check of this kind for a set of values of the letters taken at random gives a misleading result. If the student should check with two sets of values of the letters the chances of error would be still further diminished. But this is in general unnecessary.

EXERCISES

Add the following polynomials and check your results:

1. $2a + 3b + 6c$, $7a + b + c$.
2. $x + y + z + w$, $2x + y + 3z + 16w$, $x + 2z$, $3y + 10w$.
3. $8ac + 2bc + 3c + 8b$, $2b + c + bc + 7ac$, $4b + c + bc + 3ac$, $7bc + ac + c$, $3b + c$.
4. $4a + 7m + 3n$, $5ab + 3m + cn$, $2ab + 4m + 3cn$.
5. $6x + 3y + 8t + 17w$, $x + 5y + 16t$, $15x + 14y + 3t$.

6. $13a^3+3a^2+6a+1,\ 2a^3+3a,\ 5a^2+a+2,$
$16a^3+20a^2+21a+30.$

7. $x^4+x^3+x^2+x+1,\ 5x^4+2x^2+3,\ 4x^4+3x^3+3x^2+x+4.$

8. $\frac{1}{2}a+\frac{2}{3}b+\frac{2}{5}c,\ a+b+2c,\ \frac{1}{3}a+\frac{1}{2}b.$

9. $2\frac{1}{2}x^2y^3+4.2x^3y^2,\ 1\frac{3}{4}x^2y^3+\frac{1}{3}x^3y^2,\ 4.3x^3y^2+6x^2y^3.$

10. $4m^5n^3+3m^4n^4+5m^3n^5,\ 13m^5n^3+15m^4n^4+3m^3n^5,$
$4m^5n^3+3m^4n^4+2m^3n^5.$

11. $3abc,\ 15b^2c,\ 12b^3c^2,\ 9abc+10b^2c+15b^3c^2,\ 3b^2c^3+3b^3c^2.$

12. $3a^3+.56a^2b+.42ab^2+b^3,\ 7a^3+.75b^3,$
$.65a^3+.46a^2b+.78ab^2+4b^3.$

13. $mn^4+4mn^3+5mn^2+2mn+1,\ mn^3+8mn^2+mn+3,$
$mn^4+5mn^3+3mn^2+3mn.$

14. $6a^4bc^2+7a^3b^2c^3+8a^2b^3c^4+2ab^4c^5,\ 18a^3b^2c^3+2a^2b^3c^4,$
$13a^3b^2c^3+10ab^4c^5,\ 38a^3b^2c^3+2ab^4c^5.$

15. $4a^3+7a^2b+5ab^2+16b^3,\ 2a^3+3ab^2,$
$9a^3+2a^2b+6ab^2+b^3,\ 3a^3+2a^2b+10ab^2+5b^3,$
$8a^3+7a^2b+4ab^2+22b^3.$

16. $a^4+4a^3+6a^2+4a+1,\ a^3+3a^2+3a+1,$
$a^2+2a+1,\ a+1.$

17. $r+2s,\ s+2t,\ t+2r.$

18. $x^3y^3z^3+x^2y^2z^2+xyz+1,\ x^4y^4z^4+1.$

19. $12a^2bc^3,\ 21a^3b^2c,\ 25ab^3c^2,\ 4.$

20. $5x^2yz,\ 8x^3,\ 10y^3,\ 4z^3,\ 14x^3+8x^2yz+10y^3+4z^3+3yz^2.$

21. $a+b+c+d,\ x+y+z+w,\ a+2x,\ 5b+3y,\ 7c+5z,$
$10d+4w.$

22. $\frac{1}{2}x+\frac{1}{4}y+\frac{1}{3}z,\ \frac{3}{2}x+\frac{5}{4}y+\frac{5}{3}z,\ 2x+\frac{1}{2}y+3z.$

23. $l+m,\ m+n,\ n+l.$

24. $x^5+5x^4+10x^3+10x^2+5x+1,\ 2x^4+8x^3+12x^2+8x+2,$
$5x^2+10x+5.$

25. $2.5a^3+7.6b^3,\ 4a^3+6.5a^2b+9.4ab^2+2.3b^3,\ .5a^3+.8b^3.$

26. $3a^x + 4b^y + 6c^z,\ 10a^x + 5c^z,\ 17b^y + 17c^z.$

27. $12a^rb^s + 15a^t,\ 31a^rb^s + 27a^t,\ 19a^rb^s.$

28. $2(a+b) + 5(x+y),\ 24(a+b) + 7(x+y) + 10.$

29. $10(a+b+c) + 3(x+y+z),\ (a+b+c) + 16(x+y+z),$
$16(a+b+c) + (x+y+z).$

30. $5x^2 + 6xy + 2(y+z),\ 8x^2 + 12(y+z),\ 18(y+z) + 3xy.$

Simplify the following polynomials by combining the similar terms:

31. $4a^3 + 9a^2b + 6ab^2 + 3a^3 + 7b^3 + 13ab^2 + 3b^3 + 9a^2b.$

32. $7x + 5y + 4x + 10z + 8y + 17z + 250x.$

33. $a^2bx^3 + ac^2y^4 + 25a^2bx^3 + 11b^3cy^2 + 26ac^2y^4 + 9b^3cy^2.$

34. $l^3 + m^3 + n^3 + 5lmn + 4l^3 + 3lmn + 8m^3 + lmn.$

35. $a^3 + 3a^2b + 3ab^2 + b^3 + a^3 + 2a^2b + ab^2 + a^3 + a^2b.$

SUBTRACTION

20. If we subtract three times a number from seven times the same number, we shall have left four times that number; or, in algebraic terms, $7x - 3x = 4x$. The general principle involved in this statement is as follows:

Principle. — *The difference between two monomials that contain the same literal factors is obtained by multiplying the product of these factors by the difference of the coefficients of this product in the respective monomials. If neither of the monomials contains a literal factor the difference is found as in arithmetic.*

Example. The result of subtracting $5x^2y$ from $12x^2y$ is $7x^2y$, and this is expressed in symbols thus:

$$12x^2y - 5x^2y = 7x^2y.$$

21. Subtrahend, Minuend, and Difference. — The number or expression that is subtracted is called the **subtrahend**, the number or expression from which the subtrahend is subtracted is called the **minuend**, and the result obtained by subtracting the subtrahend from the minuend is called the **difference**, or **remainder.**

The work of subtraction may be arranged as indicated:

$$\begin{array}{rl} 12\,x^2y & \textit{Minuend.} \\ 5\,x^2y & \textit{Subtrahend.} \\ \hline 7\,x^2y & \textit{Difference.} \end{array}$$

It is not necessary to write the words minuend, subtrahend, and difference.

EXERCISES

Subtract the second monomial from the first in each of the following cases:

1. $14\,ab$, $9\,ab$.
2. $22\,mnxy$, $mnxy$.
3. $4\,xyz$, $3\,xyz$.
4. $3\,abcxy$, $3\,abcxy$.
5. $2\,xy$, $\frac{1}{2}\,xy$.
6. $\frac{7}{8}\,mnxy$, $\frac{1}{4}\,mnxy$.
7. $171\,mn$, $24\,mn$.
8. $27\,abx$, $25\,abx$.
9. $38\,xyz$, $37\,xyz$.
10. $14\,bcd$, $14\,bcd$.
11. $7\,a^3b^2$, $4\,a^3b^2$.
12. $25\,m^3n^2xy^3$, $3\,m^3n^2xy^3$.
13. a^5b^3, a^5b^3.
14. $20\,a^2bc$, $9\,a^2bc$.
15. $3\frac{1}{2}\,x^2y^3$, $3\,x^2y^3$.
16. $467\,a^3x^2$, $124\,a^3x^2$.
17. $21\,b^5m^3n^2$, $16\,b^5m^3n^2$.
18. $5\,r^4s^3$, $4\,r^4s^3$.
19. $x^4y^5z^5$, $x^4y^5z^5$.
20. $b^4m^4x^3$, $.9\,b^4m^4x^3$.
21. $7\,A^2B^3$, A^2B^3.
22. $28\,RST$, $27\,RST$.
23. $154\,b^7c^{10}d$, $49\,b^7c^{10}d$.
24. $22\,y^4$, $3\,y^4$.
25. $15\,r^3s^2t^3$, $15\,r^3s^2t^3$.
26. $27\,a^xb^y$, $5\,a^xb^y$.
27. $14\,a^x$, $14\,a^x$.
28. $21\,l^xm^yn^z$, $4\,l^xm^yn^z$.
29. $144\,x^ay^b$, $143\,x^ay^b$.
30. $9\,a^xb^xc^x$, $3\,a^xb^xc^x$.

31. $5(b+c)$, $2(b+c)$.

32. $14(x+y+z)$, $9(x+y+z)$.

33. $7(a^2+b^2+c^2)$, $6(a^2+b^2+c^2)$.

34. $28(x^3+y^3)$, $28(x^3+y^3)$.

35. $(a+b+x+y)$, $(a+b+x+y)$.

22. Subtraction of Polynomials. — The rule for the addition of polynomials is based on the fact that we can add to a given number the sum of two or more numbers by adding these numbers successively to the given number. In a similar way we can subtract the sum of two or more numbers from a given number by subtracting these numbers successively from the minuend.

For example, the result of subtracting $7a^3b^2+3a^2b^3+5b^3$ from $9a^3b^2+8a^2b^3+6b^3$ is the same as the result of subtracting $7a^3b^2$, $3a^2b^3$, and $5b^3$ successively from $9a^3b^2+8a^2b^3+6b^3$ and is therefore equal to $2a^3b^2+5a^2b^3+b^3$.

In finding the difference between two polynomials it is convenient to proceed in accordance with the following

Rule. — *Write the subtrahend under the minuend, placing each term under a similar term.*

Subtract each term of the subtrahend from the term above it in the minuend in accordance with § 20 and form the polynomial whose terms are the remainders thus obtained.

Example. Subtract

$$5x^3+4x^2+3 \text{ from } 2x^4+8x^3+4x^2+6x+5.$$

$$\begin{array}{lrrrrr}
 & 2x^4 & +8x^3 & +4x^2 & +6x & +5 \\
 & & 5x^3 & +4x^2 & & +3 \\
\hline
\textit{Difference.} & 2x^4 & +3x^3 & + & 6x & +2
\end{array}$$

This rule cannot be used if the subtrahend contains a term that is not similar to some term of the minuend. In such a case it is necessary to proceed as explained in § 37.

The student should devise a rule for checking the work of subtraction. Compare § 19.

EXERCISES

In each of the following exercises subtract the second polynomial from the first and check your work:

1. $12a + 3b + 6c,\ 7a + b + c.$

2. $2x + y + 3z + 16w,\ x + y + z + w.$

3. $8ac + 2bc + 3c + 8b,\ 2b + c + bc + 7ac.$

4. $13a^3 + 3a^2 + 6a + 1,\ 2a^3 + 3a.$

5. $4x^4 + x^3 + 3x^2 + x + 4,\ 4x^4 + x^3 + 3x^2 + x + 4.$

6. $4m^5n^3 + 3m^4n^4 + 5m^3n^5,\ 4m^5n^3 + 3m^4n^4 + 5m^3n^5.$

7. $14a^3 + 7a^2b + 5ab^2 + 26b^3,\ 8a^3 + 7a^2b + 4ab^2 + 22b^3.$

8 $2\frac{1}{2}x^2y^3 + 4.2x^3y^2,\ .3x^3y^2 + .6x^2y^3.$

9. $.6x + .3y + .8t + 1.7w,\ .2x + .3y + .8t + .17w.$

10. $6a^4bc^2 + 7a^3b^2c^3 + 8a^2b^3c^4 + 2ab^4c^5 + 8,\quad 6a^4bc^2 + 7a^3b^2c^3 + 8a^2b^3c^4 + 2ab^4c^5.$

11. $\frac{1}{2}a + \frac{2}{3}b + \frac{2}{5}c,\ \frac{1}{3}a + \frac{1}{2}b.$

12. $6x^3y + 4x^2y^2 + 3xy^3 + y^3,\ 2x^3y + 4x^2y^2 + y^3.$

13. $a^3b^3 + 3a^2b^2c + 3abc^2 + c^3,\ a^3b^3 + c^3.$

14. $a^3 + 6a^2 + 12a + 8,\ a^3 + 3a^2 + 3a + 1.$

15. $.5x^4 + 7x^3y + .14x^2y^2 + 3xy^3 + 9y^4,\quad .4x^4 + .6x^3y + .13x^2y^2 + 2xy^3 + 8y^4.$

16. $7a^3b + .8b^4,\ .7a^3b + .7b^4.$

17. $x + \frac{1}{2}y + \frac{1}{4}z,\ \frac{1}{2}x + \frac{1}{4}y + \frac{1}{8}z.$

18. $x^2 + y^2 + 5z^2,\ x^2 + y^2 + z^2.$

19. $a^4 + 2a^2b^2 + b^4,\ a^4 + a^2b^2 + b^4.$

20. $9 + 6a + a^2,\ 4 + 4a + a^2.$

21. $4x^3 + 4y^3 + 3x^2y + 3xy^2,\ 2x^2y + 3x^3 + 2xy^2 + 3y^3.$

22. $a^4 + 8a^3 + 2.4a^2 + 32a + 16,\ a^3 + .6a^2 + 1.2a + 8.$

23. $a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$, $a^5 + a^4b + a^3b^2 + a^2b^3 + ab^4 + b^5$.

24. $x^3 + 3x^2y + 3xy^2 + y^3$, $x^3 + 3x^2y + 3xy^2$.

25. $a^3x^2 + 5by^2 + 12cz^2$, a^3x^2.

26. $a^4 + a^2b^2 + b^4 + 2$, 1.

27. $3a^x + 7b^y + 20c^z$, $a^x + 7b^y + 19c^z$.

28. $x^my^n + y^mz^n + z^mx^n$, $x^my^n + y^mz^n + z^mx^n$.

29. $5a^m + 12b^n$, $2a^m + 3b^n$.

30. $2(x+y) + 17(y+z)$, $(x+y) + 3(y+z)$.

31. $5(a+b) + 4a^2 + 24b^2$, $3(a+b)$.

32. $x^3 + y^3 + 3(x+y)$, $x^3 + y^3 + (x+y)$.

33. $6(a+b) + 8(x+y+z) + 11$, $3(a+b) + 5(x+y+z) + 9$.

34. Subtract the sum of the polynomials in Ex. 12 of the Exercises on p. 18 from the sum of those in Ex. 15 on the same page.

CHAPTER III

NEGATIVE NUMBERS

23. First Illustration. — If the weather report for a certain day should give the temperature as 25° above zero at eight in the morning and should predict a rise of 10° for the next day, we should expect the thermometer to go to 35° above zero the next day. If the prediction should be for a drop of 10° we should expect a temperature of 15°. That is, if there is to be a rise, we add the amount of this rise to the original temperature to get the new temperature; and if there is to be a drop, we get the new temperature by subtracting the amount of this drop from the original temperature. But if the original temperature were 5° above zero and a drop of 10° should be predicted, we could not get the temperature to be expected in this way, since we have learned in arithmetic that we cannot subtract a given number from a smaller one.

Second Illustration. — If in a game of football one side gains a certain number of yards by a play and loses a smaller number in the next play, the total result of the two plays is a gain equal to the difference between the gain and the loss. But if the loss were greater than the gain we should have the same difficulty as in the case of the temperature.

Third Illustration. — If in a business transaction a man should make a profit of so many dollars and then lose a certain amount, the net result of the two transactions could be determined by subtracting the amount of the loss from the amount of the gain, provided that the loss is less than the gain. But if the loss is greater than the gain we have the same difficulty as before.

These simple and familiar examples make it clear that there are many important circumstances which our arithmetic and our algebra, as we have thus far developed it, cannot take into account. Everybody knows what the temperature will be if the thermometer drops 10° from 5° above zero, and what the effect will be on the fortunes of a football eleven of a gain of 12 yards followed by a loss of 15 yards, and what will be the net result to a man in business of two transactions in the first of which he gains $400 and in the second of which he loses $900. Moreover it is easy enough to express these things in ordinary language.

24. Usefulness of Negative Numbers in Algebra. — But in order to say these things algebraically and to be able to use our algebra to determine the information contained in statements involving facts like these it is necessary that we make our numbers express more than they have heretofore, and that we bring into use for this purpose a new kind of number. We must use numbers to express certain qualities in addition to numerical value; as, for example, the quality of being above zero in the case of the thermometer, of being in the direction of a certain goal from the center of a football field, of being a gain in a business transaction, or any one of a great many other qualities. When we have fixed our attention on a certain quality we shall say that the numbers that represent this quality are **positive** numbers and we shall indicate that a number is positive by putting the sign + in front of it and a little above the middle of it. For example, in the case of the thermometer, if we fix our attention on the quality of being above zero we shall say $^{+}17°$, instead of 17° above zero.

But, as we have seen, when we have fixed our attention on the numbers that represent a certain quality in addition to numerical value and are therefore called positive, we are under the necessity of letting other numbers represent the opposite quality. We shall call such numbers **negative** numbers and shall indicate that a number is negative by putting the sign — in front of it and a little above the middle of it. If we use positive numbers to represent degrees above zero on a thermometer, the negative numbers will represent degrees below zero. Thus we shall say $^{-}17°$ instead of 17° below zero. If we use positive numbers to represent a rise in temperature we must use negative numbers to represent a fall in temperature. If positive numbers represent distances from a line toward one side, the negative numbers will represent distances from this same line toward the opposite side. If positive numbers represent gain, negative numbers will represent loss, and so on.

Historical Note. — The Hindus in the fifth century had fairly clear notions concerning negative numbers. They thought of positive numbers as representing assets and of negative numbers as representing liabilities. But European mathematicians, for the most part, continued to use only positive numbers until about the beginning of the seventeenth century. (See historical note on p. 110.)

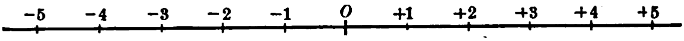
25. In arithmetic only the numerical values of numbers are considered. In that branch of mathematics there are no such things as positive numbers and negative numbers, but merely numbers. But all the results would still hold true if we considered all the numbers of arithmetic as positive. Hence, inasmuch as we have now associated with each number a certain quality in addition to its numerical value, it will be convenient for us to think of the numbers of arithmetic as positive numbers.

If we think of all the numbers of arithmetic as positive, it is easy to see why we could not subtract a number from a smaller one. But if we use also the negative numbers, which we have just described, we can subtract any number from any other one, as we shall see. (§ 31.)

If the temperature is 5° above zero, and drops 10°, what is the new temperature? If it drops 5°? If it is 12° below and drops 17°? If it is 12° above and rises 17°? If it is 12° below and rises 12°?

To refuse to take these new numbers into consideration because we are familiar only with the numbers of arithmetic, and these seem natural and easy for us, would, in the face of the facts that present themselves to us, be as absurd as it would be to graduate all our thermometers beginning with zero in order that we might not be troubled with two kinds of temperature; namely, that above zero and that below; or, as it would be for a business man to ignore his losses and to think only of his profits.

26. Positive and Negative Numbers may represent Distance and Direction. — If we consider a straight line with a point 0 on this line and a given distance for the unit distance, the numbers of arithmetic represent the distances from 0 of points



on this line. But there are two points on the line at any given distance, as 5 units, from 0, one to the right of 0 and one on the left; and the numbers of arithmetic therefore cannot represent all the points of the line. But if we make use of the positive and negative numbers that have just been described, we can represent the points on one side of 0 (the right side, for example), by the positive numbers, and the points on the opposite side of 0 by the negative numbers. These numbers then represent both distance and direction.

Where are the points represented by the following numbers:

$$^{+}2,\ ^{+}4,\ ^{-}5,\ 0,\ ^{-}3,\ ^{-}11,\ ^{+}4?$$

27. Absolute Value of Numbers. — Since the points represented by the numbers $^{+}4$ and $^{-}4$ are equidistant from the origin, we say that these numbers have the same **absolute value.** In general, we say that the absolute value of a positive number is the number itself, and that the absolute value of a negative number is the number with its sign changed.

Thus the absolute value of $^{+}5$ is 5 and the absolute value of $^{-}7$ is 7.

What is the absolute value of each of the following numbers:

$$^{+}4,\ ^{+}2,\ ^{-}5,\ 0,\ ^{+}3,\ ^{-}2,\ ^{+}5,\ _{-}3,\ _{+}3,\ ^{-}4\,?$$

Which of these numbers have the same absolute value?

28. We naturally expect to get the total change of a thermometer for two days by adding the changes for each of the days. Thus, if there should be a rise of 7° one day and one of 10° the next, the final position of the mercury would be 17° above the original position. We express this by saying $^{+}7 + {}^{+}10 = {}^{+}17$. If there should be a drop of 10° the second day, the final position of the mercury would be 3° below the original one. We express this by saying $^{+}7 + {}^{-}10 = {}^{-}3$. If there should be a drop of 7° the first day and one of 10° the second day, the mercury would finally be 17° below its original position. We express this by saying $^{-}7 + {}^{-}10 = {}^{-}17$. But if there should be a rise of 10° the second day, the final temperature would be 3° above the original one. We express this by saying $^{-}7 + {}^{+}10 = {}^{+}3$. If there is a drop of 7° the first day and a rise of 7° the next day, the final temperature is the same as the original one. We express this by saying $^{-}7 + {}^{+}7 = 0$.

These examples illustrate the following new

Principle. — *The sum of two numbers of like sign is the sum of the absolute values of the two numbers with the same sign that the two numbers have.*

The sum of two numbers with unlike signs is the difference of their absolute values with the sign of the one that has the greater absolute value. If the absolute values are equal the sum is zero.

29. Algebraic Sum. — If in getting the sum of several numbers we take into account their signs we call the result the **algebraic sum** of the numbers.

Hereafter in this book the word sum will be used in the sense of algebraic sum. Sometimes for the sake of emphasis we shall use the expression algebraic sum.

30. Motion may be represented by Positive and Negative Numbers. — If a point moves on a line we can describe the motion by telling how far the point has moved and in what direction. If the line is horizontal it will be desirable, in view of the agreement just made, to represent motion to the right by positive numbers and motion to the left by negative numbers.

If a point starts at 0 and moves along the line 7 units to the right, then 10 units to the left, then 4 units to the right, then 2 units to the right, and then 8 units to the left, its final position is 5 units to the left of 0, since

$${}^{+}7 + {}^{-}10 + {}^{+}4 + {}^{+}2 + {}^{-}8 = {}^{-}5.$$

The principle stated in § 13 concerning the sum of several monomials holds, regardless of the signs of the numerical coefficients of these monomials.

EXERCISES AND PROBLEMS

Perform the following indicated additions:

1. ${}^{+}9 + {}^{+}6$.
2. ${}^{-}15 + {}^{+}22$.
3. ${}^{-}4 + {}^{+}2$.
4. ${}^{-}10 + {}^{-}33$.
5. ${}^{-}12\frac{2}{3} + {}^{+}6\frac{5}{6}$.
6. ${}^{+}24 + {}^{+}17 + {}^{-}32 + {}^{-}50$.
7. ${}^{-}29 + {}^{-}32 + {}^{-}3 + {}^{+}64$.
8. ${}^{+}5\,a + {}^{-}6\,a$.
9. ${}^{+}17\,x + {}^{+}3\,x + {}^{-}25\,x$.
10. ${}^{+}4\,a^2b^3c^5 + {}^{-}3\,a^2b^3c^5 + {}^{+}20\,a^2b^3c^5$.
11. ${}^{-}8\,x^3yz + {}^{-}11\,x^3yz + {}^{-}3\,x^3yz + {}^{-}10\,x^3yz$.
12. ${}^{-}10\,a^2b + {}^{+}10\,a^2b$.
13. ${}^{+}10\,a^2b + {}^{-}10\,a^2b$.
14. ${}^{-}9\,a^3x + {}^{-}12\,a^3x + {}^{-}20\,a^3x$.
15. ${}^{+}9\,a^3x + {}^{+}12\,a^3x + {}^{+}20\,a^3x$.
16. ${}^{+}25\,abxy + {}^{-}26\,abxy$.
17. ${}^{-}8\,x^2y^2 + {}^{+}9\,x^2y^2$.
18. ${}^{-}1\,a^3 + {}^{+}6\,a^3 + {}^{+}10\,a^3 + {}^{-}15\,a^3$.
19. ${}^{+}3\,m^4n^3 + {}^{+}5\,m^4n^3 + {}^{+}40\,m^4n^3$.
20. ${}^{+}29\,a^3b^2c^4 + {}^{-}25\,a^3b^2c^4$.

21. ${}^{-}25\,a^3b^2c^4 + {}^{+}29\,a^3b^2c^4$.

22. ${}^{+}1\,x^4y^4 + {}^{-}2\,x^4y^4 + {}^{+}1\,x^4y^4$.

23. ${}^{-}7\,abc + {}^{+}5\,abc + {}^{+}7\,abc + {}^{-}2\,abc$.

24. ${}^{+}3\,a^x + {}^{-}5\,a^x + {}^{-}8\,a^x + {}^{-}3\,a^x$.

25. ${}^{-}10\,a^xb^yc^z + {}^{-}4\,a^xb^yc^z + {}^{-}5\,a^xb^yc^z$.

26. ${}^{+}40\,x^my^n + {}^{+}26\,x^my^n + {}^{+}30\,x^my^n$.

27. ${}^{+}5(a+b) + {}^{-}7(a+b) + {}^{+}14(a+b)$.

28. ${}^{-}14(x+y+z) + {}^{+}12(x+y+z) + {}^{-}19(x+y+z)$.

29. ${}^{-}3(a+x) + {}^{-}5(a+x) + {}^{-}6(a+x)$.

30. ${}^{+}5(a+b) + {}^{+}20(a+b) + {}^{+}41(a+b)$.

31. ${}^{+}1(b+c) + {}^{-}3(b+c) + {}^{+}2(b+c)$.

By the *average* of a series of n numbers we mean the sum of these numbers divided by n.

In solving the following problems, use both positive and negative numbers.

32. Eight men have an average weight of 160 pounds. What would be the total weight registered by them when holding down a balloon that has a lifting power of 400 pounds?

33. If a train is going at the rate of 30 miles an hour and a passenger on it is walking toward the rear coach at the rate of 176 feet per minute, how fast is the passenger moving in the direction the train is going?

Consider time after noon as positive and time before noon negative in Problems 34 and 35.

34. What is the time midway between 11 A.M. and 3 P.M.?

35. Four men were to meet at a designated place at noon. The first three arrived at 11:50, 11:57, and 11:58 respectively, but the fourth one was 20 minutes late. What was their average time of arrival?

36. At 8 A.M. on a certain day the temperature was 8° above zero. It then rose at the rate of 2° an hour until 2 P.M. After

this time it went down at the rate of 3° an hour. What was the temperature at midnight?

37. If a point starts at 0 and moves along a line 3 units to the left, then 5 units to the left, and then 7 units to the right, what is its final position?

38. If a point starts at 0 and moves 10 units to the left, then 4 units to the right, then 5 units to the right, and then 3 units to the left, what is its final position?

39. If a point starts at 0 and moves 24 units to the right, then 30 units to the left, and then 6 units to the right, what is its final position?

40. If a point starts at 0 and moves 12 units to the left, then 4 units to the left, and then 2 units to the left, what is its final position?

41. If a point starts at 0 and moves 14 units to the right, then 10 units to the right, and then 24 units to the left, what is its final position?

SUBTRACTION

31. Definition of Subtraction. — To subtract a first number from a second number is to find a third one such that when it is added to the first one the sum equals the second one.

Thus, $^{+}10 - {}^{+}3 = {}^{+}7$, because $^{+}7 + {}^{+}3 = {}^{+}10$. If we wish to subtract $^{-}3$ from $^{+}10$, that is, to find the value of $^{+}10 - {}^{-}3$, we consider that we must add $^{+}3$ to $^{-}3$ to get zero (see § 28), and that we must add $^{+}10$ to zero to get $^{+}10$. Hence $^{+}10 - {}^{-}3 = {}^{+}3 + {}^{+}10$ or $^{+}10 + {}^{+}3$. If we wish to subtract $^{-}3$ from $^{-}10$, we observe that $^{-}3 + {}^{+}3 = 0$ and that $0 + {}^{-}10 = {}^{-}10$. Hence $^{-}10 - {}^{-}3 = {}^{+}3 + {}^{-}10 = {}^{-}10 + {}^{+}3 = {}^{-}7$. If we wish to subtract $^{+}3$ from $^{-}10$, we observe that $^{+}3 + {}^{-}3 = 0$ and that $0 + {}^{-}10 = {}^{-}10$. Hence $^{-}10 - {}^{+}3 = {}^{-}3 + {}^{-}10 = {}^{-}10 + {}^{-}3 = {}^{-}13$.

These examples illustrate the following general

Principle. — *The difference between two numbers is equal to the algebraic sum of the minuend and the subtrahend with its sign changed.*

EXERCISES

Perform the following indicated subtractions:

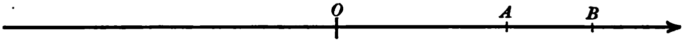
1. ${}^{+}5 - {}^{-}3.$
2. ${}^{-}5 - {}^{+}3.$
3. ${}^{-}5 - {}^{-}3.$
4. ${}^{+}17 - {}^{+}17.$
5. ${}^{+}29 - {}^{+}42.$
6. ${}^{+}34 - {}^{+}25.$
7. ${}^{-}18 - {}^{+}26.$
8. ${}^{+}22 - 0.$
9. ${}^{+}9\,abc - {}^{+}14\,abc.$
10. ${}^{+}21\,xyz - {}^{-}21\,xyz.$
11. ${}^{+}1\,m^4n^3 - {}^{+}5\,m^4n^3.$
12. ${}^{+}28\,amx - {}^{+}1\,amx.$
13. ${}^{+}5\,b^4c^4 - {}^{+}4\,b^4c^4.$
14. ${}^{+}\frac{16}{3}\,a^3b^3x^4y^2 - {}^{-}\frac{5}{6}\,a^3b^3x^4y^2.$
15. ${}^{+}4\frac{3}{5}\,x^5y^6 - {}^{+}5\frac{1}{4}\,x^5y^6.$
16. ${}^{-}14\,a^4b^4 - {}^{-}14\,a^4b^4.$
17. ${}^{+}14\,a^4b^4 - {}^{-}14\,a^4b^4.$
18. ${}^{+}26\,x^3y^2 - {}^{+}25\,x^3y^2.$
19. ${}^{+}25\,x^3y^2 - {}^{+}26\,x^3y^2.$
20. ${}^{+}9\,a^3b^3x^3 - {}^{+}8\,a^3b^3x^3.$
21. ${}^{+}9\,a^3b^3x^3 - {}^{-}8\,a^3b^3x^3.$
22. ${}^{+}30\,m^4n - {}^{-}30\,m^4n.$
23. ${}^{-}5\,a^3c^3m^3 - {}^{-}40\,a^3c^3m^3.$
24. ${}^{-}20\,m^3x^4 - {}^{+}21\,m^3x^4.$
25. ${}^{-}13\,b^3 - {}^{+}13\,b^3.$
26. ${}^{+}12\,a^x - {}^{-}12\,a^x.$
27. ${}^{-}7\,a^mb^n - {}^{+}14\,a^mb^n.$
28. ${}^{+}22(m+n) - {}^{-}40(m+n).$
29. ${}^{-}19(a^2+b^2) - {}^{-}3(a^2+b^2).$
30. ${}^{-}4(a+b+x+y) - {}^{+}8(a+b+x+y).$
31. ${}^{+}10(x+y) - {}^{+}12(x+y).$
32. ${}^{-}14(b+c) - {}^{-}15(b+c).$

SEGMENTS OF A LINE

32. Definitions. — Consider a line of unlimited length, which we shall call an **axis**, and a given direction (indicated by an arrow) on this line as the positive direction.

We shall call the part of this line from one point A to another B a **segment** and shall say that the segment is positive if B is in the direction of the arrow from A, and negative if B is in the direction opposite to that of the arrow from A.

We shall call this segment AB. It should be noticed that the segments AB and BA are not the same, inasmuch as one is positive and the other is negative.



We have already seen (§ 26) that if we have a starting point (or origin) as 0, there is a certain number that represents the distance of A from 0.

33. Abscissa. — The distance of A from 0 is called the **abscissa** of A with respect to the origin 0.*

It is obvious that if we add the distance from A to B (taken with the proper sign) to the abscissa of A the sum will be the abscissa of B. *Hence the distance from A to B is equal to the abscissa of B minus the abscissa A.*

If, for example, the abscissæ of A and B are $^{+}3$ and $^{+}8$ respectively, then $AB = {}^{+}8 - {}^{+}3 = {}^{+}5$.

What does BA equal?

EXERCISES

Find AB and BA when:

1. $A = {}^{+}2,\ B = {}^{+}7.$
2. $A = 0,\ B = {}^{-}8.$
3. $A = {}^{+}2,\ B = {}^{-}7.$
4. $A = {}^{+}9,\ B = {}^{+}4.$
5. $A = {}^{-}12,\ B = {}^{+}5.$
6. $A = {}^{-}6,\ B = {}^{+}8.$
7. $A = {}^{-}3,\ B = {}^{-}1.$
8. $A = {}^{-}\frac{3}{4},\ B = {}^{+}4.$
9. $A = {}^{+}4,\ B = {}^{-}4.$
10. $A = {}^{-}1,\ B = {}^{-}1.$
11. $A = {}^{+}10,\ B = {}^{-}2.$
12. $A = {}^{-}2,\ B = {}^{+}10.$
13. $A = 0,\ B = {}^{+}5.$
14. $A = {}^{+}12,\ B = {}^{+}12\frac{1}{2}.$
15. $A = {}^{+}4a,\ B = {}^{+}3a.$
16. $A = {}^{-}4a,\ B = {}^{+}3a.$
17. $A = {}^{+}7x,\ B = 0.$
18. $A = {}^{-}5x,\ B = {}^{+}5x.$
19. $A = {}^{+}.5x,\ B = {}^{-}2.5x.$
20. $A = {}^{-}10x,\ B = {}^{-}9x.$

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* Compare § 76.

21. The abscissæ of the points *A*, *B*, *C*, and *D* are $^{+}7$, $^{-}5$, $^{-}2$, and $^{+}8$ respectively. What are the segments *DA*, *DB*, and *DC*?

22. The abscissæ of *A* and *B* are $^{+}4$ and $^{+}10$ respectively. What are the distances of these points from the point whose abscissa is $^{+}7$?

23. If the abscissæ of *A*, *B*, *C*, and *D* are $^{-}2$, $^{+}5$, $^{+}6$, and $^{-}8$ respectively, what are the segments *AB*, *AC*, *AD*, *BC*, *BD*, and *CD*?

24. What is the sum of the segments *AB*, *BC*, *CD*, and *DA* of the preceding exercise?

The average of the abscissæ of two points is the abscissa of the point midway between them. (See Ex. 93, p. 77.)

25. What is the abscissa of the point midway between *A* and *B* of Ex. 23?

26. What is the abscissa of the point midway between *B* and *C* of Ex. 23?

27. Where is the midpoint of the segment *AB* of Ex. 9?

28. The local time of high water at New Bedford, Mass., is 10 minutes earlier than it is at Governor's Island, New York Harbor; and at Nantucket, Mass., it is 4 hours and 21 minutes later than at Governor's Island. Assuming that New Bedford and Nantucket have the same local time, find the difference in the time of high water at the two places.

29. The time of high water at Albany, N.Y., is 9 hours and 31 minutes later than it is at Governor's Island, while that at Atlantic City, N.J., is 20 minutes earlier than that at Governor's Island. What is the average time of high water at Albany and Atlantic City as compared with that at Governor's Island?

30. In the New York World Almanac for 1913, p. 75, the following directions are given in connection with tables showing the times of high tide at various places: To find the approximate time of high tide at Atlantic City, N.J., on any day, find first the time of high water at New York under the desired date, and then subtract 20 minutes.

The time of high water at New York for the afternoon of Feb. 13, 1913, is given as 12:8. What was the corresponding time of high water at Atlantic City?

When it is noon at Greenwich, it is approximately 51 minutes after 11 at Paris, 7 o'clock at Denver, and 50 minutes after 5 at Chicago.

31. What is the difference in time between Paris and Denver?

32. What is the difference in time between Paris and Chicago?

34. Double Use of the Signs + and –.—If the student has thought carefully about what has been said it must have seemed to him unfortunate that we should use the signs + and – for two distinct purposes; namely, to indicate addition and subtraction on the one hand and on the other hand to designate certain qualities. It turns out, however, that this double use of these signs causes no ambiguity, and even adds to the simplicity of algebra. The following considerations will make this plain.

Our discussion of addition and subtraction shows that $a + {}^{-}b$ and $a - {}^{+}b$ have the same value. We can indicate this value by $a - b$. Likewise $a + {}^{+}b$ and $a - {}^{-}b$ are the same and we indicate their value by $a + b$. In general whenever we have to combine several monomials by the operations of addition and subtraction, if instead of adding a negative monomial we subtract the corresponding positive one, and instead of subtracting a negative monomial we add the corresponding positive one, we shall not affect the result. In considering such expressions we shall always assume that this has been done. Accordingly, the + and – signs that occur in such expressions will be looked upon as signs of operation and the monomials themselves will be taken as positive, with the exception of the first one, and the sign before it will be taken as a sign of quality. When this sign of quality is +, it is usually not written. With this understanding we can do away with the double use of the signs + and −.

For example, ${}^{-}4\,a^3b^2 + {}^{-}7\,a^2b^3 - {}^{-}3\,ab^4 - {}^{+}8\,a^4b$ will be written $-4\,a^3b^2 - 7\,a^2b^3 + 3\,ab^4 - 8\,a^4b$.

EXERCISES

Rewrite each of the following, using only one set of signs; then combine into a single monomial:

1. ${}^{-}10 - {}^{-}4 + {}^{+}7 - {}^{+}9$.
2. ${}^{+}3 + {}^{-}5 - {}^{+}17 + {}^{+}8$.
3. ${}^{+}3\,a^2b^3 + {}^{+}14\,a^2b^3 - {}^{+}5\,a^2b^3$.
4. ${}^{+}9\,xy^4z - {}^{+}4\,xy^4z - {}^{-}10\,xy^4z$.
5. ${}^{+}7\,m^3n^2 + {}^{+}16\,m^3n^2 - {}^{+}8\,m^3n^2$.
6. ${}^{+}7\,m^2n^3 - {}^{-}16\,m^2n^3 - {}^{-}8\,m^2n^3$.
7. ${}^{-}1\,x^3y^3z^3 - {}^{-}6\,x^3y^3z^3 - {}^{+}3\,x^3y^3z^3$.
8. ${}^{+}1a^3b^3 + {}^{-}3\,a^3b^3 + {}^{+}3\,a^3b^3 + {}^{-}1\,a^3b^3$.
9. ${}^{+}5\,abc - {}^{-}8\,abc + {}^{+}11\,abc$.
10. ${}^{-}7\,a^3x^3 - {}^{-}3\,a^3x^3 - {}^{-}5\,a^3x^3 + {}^{-}2\,a^3x^3$.
11. ${}^{+}10\,m^2x^2y - {}^{+}7\,m^2x^2y + {}^{-}3\,m^2x^2y$.
12. ${}^{-}30\,a^4 + {}^{-}2\,a^4 + {}^{+}10\,a^4$.
13. ${}^{+}30\,a^4 - {}^{+}2\,a^4 + {}^{+}10\,a^4$.
14. ${}^{+}1\,b^4c^3 + {}^{-}1\,b^4c^3 + {}^{+}10\,b^4c^3$.
15. ${}^{-}24\,x^4y^4 + {}^{-}6\,x^4y^4 - {}^{+}6\,x^4y^4$.

Rewrite each of the following, using only one set of signs; then simplify:

16. ${}^{+}1\,a^4 + {}^{-}4\,a^3b + {}^{+}6\,a^2b^2 + {}^{-}4\,ab^3 + {}^{+}1\,b^4$.
17. ${}^{+}3\,a + {}^{+}4\,b + {}^{-}3\,c + {}^{+}5\,a - {}^{+}2\,b + {}^{-}1\,c$.
18. ${}^{-}5\,b - {}^{+}8\,x + {}^{-}6\,y - {}^{+}13\,b + {}^{+}14\,y - {}^{-}15\,b$.
19. ${}^{+}7\,mn^2 - {}^{-}3\,m^2n - {}^{-}4\,mn - {}^{-}2\,m^2n + {}^{-}6\,mn^2$.
20. ${}^{+}12\,a^3b - {}^{+}14\,ab^3 + {}^{-}9\,b^4$.
21. ${}^{-}4\,x^2 + {}^{+}2\,xy - {}^{+}3\,y^2 - {}^{-}2\,x^2 + {}^{+}5\,xy$.
22. ${}^{-}12\,a^3 - {}^{+}5\,b^3 + {}^{+}7\,abc$.
23. ${}^{-}3\,r^2s + {}^{+}10\,rs^2 - {}^{+}4\,s^3 + {}^{-}6\,r^2s$.
24. ${}^{+}18 + {}^{+}3\,m + {}^{-}6\,n + {}^{+}7\,m - {}^{-}5 + {}^{+}2\,n$.
25. ${}^{+}10 - {}^{-}4 + {}^{-}8\,a + {}^{+}6\,b - {}^{-}5\,c - {}^{+}6\,b$.
26. ${}^{+}5\,x - {}^{-}4\,y - {}^{+}10\,z$.
27. ${}^{-}12\,a^2 + {}^{-}8\,b^2 - {}^{-}17\,c^2 - {}^{-}8\,b^2 - {}^{+}17\,c^2 + {}^{+}12\,a^2$.
28. ${}^{-}26\,a^x - {}^{-}26\,b^x - {}^{-}26\,c^x$.
29. ${}^{+}8(a+b) - {}^{-}4(x+y) + {}^{-}7(a+b)$.
30. ${}^{-}13(a+x) + {}^{+}14\;b+y) - {}^{+}8(b+y) + {}^{+}13(a+x) + {}^{-}6(b+y)$.
31. ${}^{-}4(a+x+y) - {}^{+}5(b+z) + {}^{+}8(a+x+y)$.
32. ${}^{-}8(m+n) - {}^{+}4(m+n) + {}^{+}10(m+n)^2 + {}^{-}9(m+n)^2$.
33. ${}^{+}1(r+s) - {}^{-}4(r+s)^2 + {}^{-}4(r+s)^2 + {}^{-}1(r+s)$.

35. In § 16 we defined a polynomial as the indicated sum of two or more monomials. It might seem from the way this definition is worded that such an expression as $7\,x - 3\,x$, which occurs in § 20, is not a polynomial, since it is the indicated difference of the two monomials. But we know from the principle of § 28 that $7\,x - 3\,x$ is equivalent to the sum of $7\,x$ and $-3\,x$, and it is therefore proper to think of it as a polynomial. A similar remark applies to more complicated expressions, as, for example, $-x^4y^2 - 5\,xy^3 - 8\,abc + 3\,b^2c^3$, which is equivalent to the indicated sum of the monomials $-x^4y^2$, $-5\,xy^3$, $-8\,abc$, and $3\,b^2c^3$. Indeed, it is common in algebra to include subtraction under addition because of the fact, with which the student is now familiar, that the subtraction of any number is equivalent to the addition of the negative of this number.

36. Addition. — The rule given in § 18 for finding the sum of two or more polynomials was used only in connection with polynomials all of whose terms were positive. In the light of the principle of § 28, however, we can see that the same rule is applicable to all polynomials.

For example, to get the sum of the polynomials $3\,a - 4\,b + 7\,c$, $a + b - c$, and $-2\,a - 9\,b + 14\,c$ we arrange our work as follows:

$$\begin{array}{rrrr} & 3\,a - & 4\,b + & 7\,c \\ & a + & b - & c \\ & -2\,a - & 9\,b + & 14\,c \\ \hline \textit{Sum} & 2\,a - & 12\,b + & 20\,c \end{array}$$

The result can be checked as described in § 19.

EXERCISES

Add the following polynomials and check your results:

1. $10\,a - 2\,b + c$, $-3\,a + 7\,b + 8\,c$, $15\,a - 4\,b - 8\,c$.
2. $x - y + z$, $x + y - z$, $-x + y + z$.
3. $a^3 - 3\,a^2b + 3\,ab^2 - b^3$, $a^3 + 3\,a^2b + 3\,ab^2 + b^3$, $a^3 + b^3$.
4. $4\,A - 6\,B + 2\,C$, $A + B + C$, $-10\,A - 5\,B - 3\,C$.
5. $x^3 - y^3$, $y^3 - x^3$.
6. $m+n+p+r$, $m-n-p+r$, $m+n-p-r$, $-m+n-p+r$.
7. $7\,r + 3\,s - 4\,t$, $3\,r - 2\,s + 6\,t - 5$, $-10\,r - s - 2\,t + 6$.
8. $x^3 - y^3$, $x^3 - y^3$, $y^3 - 3\,y^2x + 3\,yx^2 - x^3$.

9. $a^2 + 5a + 6,\ a^2 - 5a + 6,\ 6 - 2a^2.$

10. $6x - 3y - 8t + 17w,\ -x - 5y + 16t,\ 15x + 14y + 3t.$

11. $4m^5n^3 + 3m^4n^4 - 5m^3n^5,\quad 13m^5n^3 - 15m^4n^4 - 3m^3n^5,$
$4m^5n^3 - 3m^4n^4 + 2m^3n^5.$

12. $2\frac{1}{2}a^2b^3 - 4\frac{2}{3}a^3b^2,\ 1\frac{3}{4}a^2b^3 + \frac{1}{3}a^3b^2,\ -\frac{1}{4}a^2b^3 - \frac{1}{6}a^3b^2.$

13. $xy^4 - 4xy^3 - 5xy^2 - 2xy - 1,\ -xy^3 - 8xy^2 - xy - 3,$
$-4xy^4 + 5xy^3 + 3xy^2 + 7xy.$

14. $a^2 - 2ab + b^2,\ 2xy - x^2 - y^2.$

15. $13r^3 + 3r^2 - 6r - 1,\ -2r^3 - 3r,\ 5r^2 + r - 2,$
$-6r^3 + 20r^2 - 21r + 30.$

16. $x^4yz^2 - 7x^3y^2z^3 - 8x^2y^3z^4 - 2xy^4z^5,\quad 18x^2y^3z^4 + 2x^3y^2z^3,$
$7x^3y^2z^3 - 3xy^4z^5.$

17. $4a^2b^3c^2,\ -10ab^4c^3,\ 14b^4c^4,\ -2a^2b^3c^2.$

18. $5x^2yz - 8x^3 - 10y^3,\ -3xy^2z + x^3 - y^3,\ 4xyz^2 - x^3 + 2y^3.$

19. $a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4,\ -a^4 - b^4,$
$a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4.$

20. $l - m,\ m - n,\ n - l.$

21. $\frac{1}{2}a - \frac{1}{4}b - \frac{2}{5}c,\ a + b + c,\ \frac{3}{4}a - \frac{4}{5}c,\ \frac{3}{4}b - 2c.$

22. $r - 2s + 3t,\ 3r + s - 2t,\ -2r + 3s + t.$

23. $x^3 + y^3,\ 3x^2y - 3xy^2 - x^3 + y^3,\ -2y^3.$

24. $l^3 - m^3 + n^3,\ 2n^3 - 4l^3,\ 10m^3 + 4n^3,\ l^3 + n^3 + m^3.$

25. $x^n - y^n,\ y^n + x^n,\ x - y - x^n + y^n.$

26. $(a + b)x + 2ay,\ (a - 2b)x + (a - b)y.$

Hint. Treat $a + b$ and $a - 2b$ as if they were numerical coefficients of x, and $2a$ and $a - b$ as if they were numerical coefficients of y. Thus,

$$\begin{array}{rl} & (a + b)x + 2ay \\ & \underline{(a - 2b)x + (a - b)y} \\ \textit{Sum} & (2a - b)x + (3a - b)y \end{array}$$

27. $(a + b + c)x + (a + c)y,\ (a - b + c)x + (b + a)y,$
$(a + b - c)x + (c + b)y.$

28. $(m+n)x+(3m-n)y,\ (2m+4n)x+(m-3n)y.$

29. $(3m+n)a+(2m-n)b,\ ma+nb,\ na-mb.$

30. $(a+b)x^2+(b+c)y^2+(c+a)z^2,\ ax^2-by^2+cz^2,$ $-bx^2+cy^2-az^2.$

31. $(-a+b+c)x+(a-b+c)y+(a+b-c)z,$ $(a-b+c)x+(a+b-c)y+(-a+b+c)z,$ $(a+b-c)x+(-a+b+c)y+(a-b+c)z.$

32. $(r+s)a+(r-s)b,\ (s-r)b+(r-s)a,\ -2ra.$

33. $mx+ny+nz,\ 2mx-3ny+2mz,\ 3mx+2ny$ $+(-2m-n)z.$

34. $(l+m)a+(l-m)b,\ (l-m)a+(l+m)b.$

37. Subtraction. — When we discussed subtraction in §§ 20–22 we had not considered negative numbers and we were therefore restricted in the polynomials whose differences we could find.

State in detail what those restrictions are.

In the light however of the principle of § 31 we can now state the following rule for subtracting any polynomial from any other one.

Rule. — *Write the subtrahend under the minuend, placing each term under a similar term, if there is one similar to it.*

Reverse the signs of the terms of the subtrahend and add the resulting polynomial to the minuend.

For example, to subtract $2x^4+8x^3-4x^2+6x+5$ from $5x^3-4x^2+3$, we arrange the work as follows:

$$\begin{array}{rrrrrr} & & 5x^3 & -4x^2 & & +3 \\ & 2x^4 & +8x^3 & -4x^2 & +6x & +5 \\ \hline \textit{Difference} & -2x^4 & -3x^3 & & -6x & -2 \end{array}$$

The student should cultivate the habit of carrying the reversal of signs in his head without writing down the new signs.

The work of subtraction can be checked in the same way as the work of addition.

EXERCISES

In each of the following exercises subtract the second polynomial from the first and check your work:

1. $7a+b+c$, $12a+3b+6c$.
2. $2x-y-4z+15w$, $5x+2y+6z-8w$.
3. $-9mn+3m-2n$, $m-3mn+n$.
4. $a^4+4a^3+6a^2+4a+1$, $a^4-4a^3+6a^2-4a+1$.
5. $a+b+c$, $x-y+z$.
6. $4a^3b^5-3a^4b^4+7a^5b^3$, $5a^3b^5-7a^5b^3-3a^4b^4$.
7. $-3x+7y+8z-10w$, $x+y+z+w$.
8. $3.4a+6.3b-4.5c$, $-2.6a-5.3b+2.61c$.
9. $x^4+x^2y^2+y^4$, $x^4-2x^2y^2+y^4$.
10. $3\frac{1}{2}m+2\frac{2}{5}n+3l$, $7l-2n$.
11. $3a+5b-3c$, $3a+5b-3c$.
12. $3a+5b-3c$, $-3a-5b+3c$.
13. x^3+3x^2+3x+1, $x^3+6x^2+12x+8$.
14. r^3+3r^2+3r+2, r^3+3r^2+3r-2.
15. $7x^3y+8y^4$, $7a^3b+8b^4$.
16. $\frac{1}{3}l+\frac{1}{2}m$, $\frac{1}{2}l+\frac{2}{5}n+\frac{2}{3}m$.
17. $12A+B-C-D$, $-4A+2B-C-D$.
18. $9-6s+s^2$, $s^2-8s+16$.
19. $a^4+7a^2b^2+9b^4$, a^2b^2.
20. $x^3-3x^2y-3xy^2+y^3$, $-6xy^2-2y^3$.
21. $b^4-12b^3+54b^2-108b+81$, $b^3-9b^2+27b-27$.
22. 0, $3x-2y+5z$.
23. a^3-b^3, a^3-b^3.
24. $A+B+C+D$, $A-B+C-D$.
25. $5x^3-4x^2+3$, $-2x^4-3x^3-6x-2$.

26. $(a+b)x+(b+c)y,\ (a-b)x+(b-c)y.$

27. $(l+m+n)a+(2l-m-n)b+(m+n)c,$
$(l+m-n)a+(l+m+n)b+(m+n)c.$

28. $(2a+3b)x+(2b+3a)y,\ (a+b)x+(a+b)y.$

29. $(a+b+c)l+(2a-3b-4c)m,$
$(a-b+c)l+(2a+3b-4c)m.$

30. $(3a-7b)x^2+(5a+4b)y^2+(a-b)z^2,$
$3ax^2+4by^2+(a-6b)z^2.$

In each of the following exercises subtract the second polynomial from the first and add the subtrahend to the difference:

31. $7x-4y+6z,\ -3x-7y+10z.$

32. $10a^2+9b^2+4c^2,\ 3a^2+9b^2-5c^2.$

33. $-4a^3b^2+5ab^4-b^5,\ a^5-2a^3b^2-2b^5.$

34. $x^4+x^2y^2+y^4,\ x^4+2x^2y^2+y^4.$

35. $6l-2m-7n+1,\ -3+l-n.$

36. $-3a-4b-12c+7,\ c-2+b-3a.$

Could you have foretold the final result in each of Exs. 31–36, without going through the details of the work?

38. The Equation. — Suppose that we are told that the sum of a number and 5 is 3, and that we want to find all the numbers that answer this description. In order to bring the description clearly before us it is desirable to let the number in question be represented by some letter such as x. If we do this the description just given becomes the equation $x+5=3$. Now the number to which we must add 5 in order to get 3 for the sum is what we have called the difference between 3 and 5. Hence $x=3-5=-2$. That is, there is only one number answering the description just given and that number is -2.

When a description of a number is put in the form of an equation the number described may be represented by any letter, but the letter x is most frequently used. It is common also to use y or z.

EXERCISES

Find the numbers that answer the following descriptions:

1. $x + 5 = 6.$
2. $x - 7 = 4.$
3. $x - 21 = 0.$
4. $x + 21 = 0.$
5. $4 + x = 3.$
6. $y + 8 = -2.$
7. $9 + y = 12.$
8. $z - 13 = 15.$
9. $4 + z = 0.$
10. $2m + 5 = 12.$
11. $? + 4 = 0.$
12. $4x - 5 = 45.$
13. $x + 8 = -6.$
14. $6 + ? = -1.$
15. $? - 10 = 24.$
16. $2x - 10 = 50.$
17. $2x + 10 = 50.$
18. $7 - 8 + 5 + t = 3.$
19. $4 + 10 + l - 5 = 14.$
20. $x + 2x - 5 = 12.$
21. $x + x + 4x - 3x + 6 = 17.$
22. $x + 5x - 2x = 0.$
23. $-2x + 4x + 13 + 10x = 25.$
24. $r - 6 + 2r = 0.$
25. $2x - 3 + x = 3.$
26. $s + 3s + 5s = 27.$
27. $2y + 5y - 4y = 0.$
28. $3y - 9 + 2y = -4.$
29. $10x - 18 = 12.$
30. $7k + 12 - 2k + 1 = 13.$

Translate each of the foregoing descriptions into ordinary language.

EXERCISES AND PROBLEMS

Each of the Exercises 1–13 contains a description of a number. Represent the number by a letter, then translate this description into the form of an equation and find the number that answers the description.

1. When 7 is added to four times a number the sum is 35.

2. When 2 is subtracted from five times a number and three times the number added to the difference, the final result is 46.

3. Eight times a number exceeds five times the number by 12.

4. Four times a number exceeds the number by 15.

5. Five times a number is added to the number and twice the number is subtracted from the sum. The result is 0.

6. When 21 is added to a number the sum is 0.

7. When -8 is subtracted from a number the difference is -5.

8. The sum of 5 and three times a number is 13.

9. If -3 is added to the sum of a number and twice the number, the sum is 21.

10. A certain number is multiplied by -5, then -7 is subtracted from the product, and six times the number added to this result; the final result is 25.

11. Five times a number exceeds the number by 104.

12. The sum of four times a number and 6 exceeds the number by 21.

13. When 8 is added to nine times a number and four times the number subtracted from the result, the final result is 28.

14. In the House of Representatives of the 63d Congress there are 435 members. Of these the Democrats have a majority of 147. How many Democrats are there in the House?

15. The Republicans had a majority of 10 in the Senate of the 62d Congress, and there were 92 members altogether. How many Republican Senators were there?

16. The number of public high schools in the United States in 1911 exceeded the number of private high schools by 8255, and the total number of high schools was 12,213. How many of each kind were there?

17. The perimeter of a rectangle is 34 inches and the length exceeds the width by 5 inches. How wide is the rectangle?

18. The length of a rectangular field whose perimeter is 192 rods exceeds its width by 10 rods. What is the length?

19. What should be the length of a rectangular field that is to be twice as long as it is wide, and to have a perimeter of 150 rods?

20. The parcel post regulations state that a mailable package must not be greater in size than 72 inches in length and girth combined. A mail order firm wishes to use a box with square ends and twice as long as it is wide. What is its greatest permissible length?

21. How long a cylindrical box 40 inches in circumference can be mailed?

22. What is the greatest possible girth of a mailable package 3 feet long?

23. The time from sunrise to sunset at St. Louis on Jan. 1, 1913, was 5 hours and 2 minutes less than the time from sunset to sunrise, Jan. 2. On each day the sun rose at 7:19. At what time did it set Jan. 1?

24. A system of pensions is based on the formula

$$\frac{A}{2} + 200 = P,$$

where $A =$ active pay and $P =$ annual pension.

What must be a man's salary in order that he may retire on an annual pension of $1500?

For each of the Exercises 25–35 draw an axis, mark an origin on it, and locate each of the points mentioned.

The points A, B, and C have the abscissæ -8, -1, and 9 respectively.

25. Find the point Y such that $AB + BC + CY = 0$.

26. Find the point Z such that $AB + BZ = BC$.

27. Find the point M such that $AM + MB + CM = 0$.

28. Find the point N such that $AN = BA + \frac{1}{2}CB$.

Given the points A, B, and C with the abscissæ 5, -3, and 8 respectively.

29. Where must the point X be in order that $AX + CX = 0$?

30. Where must the point Y be in order that

$$AY + YB + CY = 0?$$

31. Where must the point Z be in order that

$$AB + BC + CZ = 0?$$

32. Find the point M such that $AM = 5\,AB$.

33. Find the point N such that $BN = 5\,BA$.

34. Find the point P such that $AP + BP + CP = 0$.

35. Find the point R such that $AR = BC - AB$.

CHAPTER IV

PARENTHESES

39. Use of Parentheses. — We indicate the difference between two monomials by writing the subtrahend after the minuend with the minus sign between them. That is, we say that the difference between $6a^2b$ and $2a^2b$ is $6a^2b - 2a^2b$. But it obviously would not be correct to say that the difference between $6a^2b$ and $2a^2b + 3ab^2 - 4b^3$ is $6a^2b - 2a^2b + 3ab^2 - 4b^3$. If, however, we could indicate in some way that for this particular occasion $2a^2b + 3ab^2 - 4b^3$ is to be looked upon as a single term (monomial), we could then indicate this difference in a concise way. This is done by inclosing $2a^2b + 3ab^2 - 4b^3$ within parentheses; and then the difference in question is indicated thus:

$$6a^2b - (2a^2b + 3ab^2 - 4b^3).$$

In general, parentheses around a polynomial indicate that the polynomial is to be looked upon as a single term.

The necessity for considering such expressions arises in the solution of many problems, of which the following is a simple illustration:

In 1913 the Electoral College cast 435 votes for Governor Wilson, 15 for President Taft, and 81 for Colonel Roosevelt. How many of the Wilson votes would have been necessary to Roosevelt in order to have given the latter a plurality of 20 over the former?

If we let $x =$ the number of votes required, we get

$$81 + x - (435 - x) = 20.$$

40. Signs of Aggregation. — It sometimes happens that we wish to inclose a polynomial such as $3x^2y - 4xy^2 + (8a^2b - 5ab^2)$

in parentheses. In such a case, in order to avoid confusion, we use a slightly different sign, such as [ ], or { }. These are called **brackets** and **braces** respectively. At times a line over the polynomial takes the place of the parentheses. Thus $6a^2b - \overline{2a^2b + 3ab^2 - 4b^3}$. This sign is called a **vinculum.**

The parentheses, the brackets, the braces, and the vinculum are called **signs of aggregation.** All the signs of aggregation are referred to as parentheses.

41. Removing Parentheses. — We can frequently simplify a polynomial by removing the parentheses contained in it. Consider, for example, the expression $81 + x - (435 - x)$, which occurs in the illustrative problem at the end of § 39. This indicates that $435 - x$ is to be subtracted from $81 + x$. If we perform this subtraction we get

$$\begin{array}{r} 81 + x \\ \underline{435 - x} \\ -354 + 2x \end{array}$$

Hence, $81 + x - (435 - x) = 81 + x - 435 + x = -354 + 2x.$

This form is simpler than the original one, and the equation in the problem referred to becomes

$$-354 + 2x = 20,$$

which is easily solved.

$3x^2y - 4xy^2 + (2x^2y + 3xy^2)$ indicates that $3x^2y - 4xy^2$ and $2x^2y + 3xy^2$ are to be added.

$$\begin{array}{r} 3x^2y - 4xy^2 \\ \underline{2x^2y + 3xy^2} \\ 5x^2y - xy^2. \end{array}$$

Hence $3x^2y - 4xy^2 + (2x^2y + 3xy^2) = 5x^2y - xy^2$. And this form is simpler than the original one.

These two examples, taken in connection with the principles of addition and subtraction which we have explained, establish this

Principle. — *Parentheses preceded by the plus sign may be removed from a polynomial without changing the signs of the terms inclosed by the parentheses. Parentheses preceded by the minus sign may be removed from a polynomial provided the sign of each term inclosed by the parentheses be changed.*

42. Parentheses within Parentheses. — If a polynomial contains parentheses within parentheses, it is best to remove one pair at a time, and the beginner will find it desirable always to remove the innermost pair first. If two pairs of parentheses occur within a third pair, but neither within the other, they may be removed simultaneously. For example,

$$\begin{aligned} &7x^3 - \{2x^2 + (7x - 5) - (8x^2 + 7x - 1)\} \\ &= 7x^3 - \{2x^2 + 7x - 5 - 8x^2 - 7x + 1\} \\ &= 7x^3 - \{-6x^2 - 4\} \\ &= 7x^3 + 6x^2 + 4. \end{aligned}$$

Use the following

Rule. — *To remove parentheses rewrite the polynomial omitting the innermost parentheses in accordance with the foregoing principle.*

Combine the similar terms that may occur within the next innermost parentheses.

Continue the process until all the parentheses have been removed and the result has been simplified.

Having in mind the principle that governs the removal of parentheses, the student should formulate the rule for the insertion of parentheses in a polynomial.

EXERCISES

Simplify the following polynomials as much as possible:

1. $5 + (7 - 3 + 2)$. **2.** $5 - (7 - 3 + 2)$.

3. $5 + 7(7 - 3 + 2) - (7 - 3 + 2)$.

4. $5x - (7x - 3x + 2x)$.

5. $4x^3y-7xy^3+8y^4-(-x^3y+3xy^3+12y^4)$.

6. $(2a+3b+6c)+(7a+b+c)$.

(Compare Ex. 1, p. 17.)

7. $(x+y+z+w)-(2x-y+3z+16w)+(x+2z)$.

8. $(4a+7m-3n)+(5ab-3m-cn)-(2ab+4m-3cn)$.

9. $[(6a^4bc^2+7a^3b^2c^3)-(8a^2b^3c^4+2ab^4c^5)]$
$-[(18a^3b^2c^3-2a^2b^3c^4)+(a^4bc^2-a^2b^3c^4)]$.

10. $(8ac+2bc+3c+8b)-(2b+c+bc+7ac)$.

(Compare Ex. 3, p. 22.)

11. $x+\{2x-(5x-3y)\}+(4x-y)$.

12. $2a+\{4b-[4c-5a-(3b-2c)+8a]+6c\}$.

13. $5a-7b-(-3a+4b)$.

14. $-(x-y-z)+(-x+y-z)-(-x-y+z)$.

15. $7x-\{5x-(2y-4y+1)\}$.

16. $-(7m^2-8mn+2n^2)-(12mn-m^2-n^2)$.

17. $4a-\overline{3a+2b}-(6x-[b+c]+3)$.

18. $(5x^2-y^2-z^2)-(3x^2-\overline{6y^2-2z^2})$.

19. $a-(b-c)+[a+(b-c)]-\{5a-[a-(b-c)]\}$.

20. $5m+(2-n)+[5-(m-2n)]-[4m+(m+n-2)]$.

21. $3a-\{a-2-(4a+6)\}$.

22. $-\{8r-(5r-3)\}+\{12r-(9r+2)\}$.

23. $5x-\overline{3x-4}-\overline{4+2x}$.

24. $2a-3b+2c-(5b-c+a)-(a+b)$.

25. $-7a^2b-3ab^2+(-6a^3+4b^3)$.

26. $-\overline{x-y}+\overline{-3x-4y}$.

27. $(3a-b-7)-(2a+b-8)+(5a+2b-3)$.

28. $9x-[6x+\{x-(3x-4)\}]$.

29. $(3m-6n)-\{4m-[4m+(5n-3m)-(7n-2m)]\}$.

30. $[5a-(3b-2c)]-[-(3b-6a)-10a]$.

31. $(a+b+c)-(a+b-c)-(a-b+c)-(-a+b+c)$.

32. $-\{5x-[2x-(4x+3b)+6x]-(8b-3)\}$.

In the following polynomials inclose all the terms containing x^2 in parentheses preceded by the plus sign, and inclose the terms containing x in parentheses preceded by the minus sign:

33. $4x^2+ax^2-3x+bx+5$.

34. $x^2+2ax+mx^2-2x+3+kx$.

35. $x^2+bx+cx+1$.

36. $x^2+2a^2b^2-a^2x^2-2b^2x$.

37. $x^2-ax-bx-2x-5$.

38. The sum of two numbers is a and the first one is x. What is the result of subtracting the second one from the first?

39. Find the two numbers whose sum is 50 and whose difference is 10.

40. What is the result of subtracting the second number of Ex. 38 from four times the first one?

41. The sum of two numbers is 52, and the result of subtracting the larger one from four times the smaller one is 8. What are the numbers?

42. The sum of two numbers is $a+b$ and the smaller one is x. What is the result of subtracting the larger one from four times the smaller one?

Further problems involving parentheses will be found in Chapter V.

CHAPTER V

MULTIPLICATION AND DIVISION

43. Multiplication of Negative Numbers. — The student has learned in arithmetic how to perform the operations of addition, subtraction, multiplication, and division upon what we now call positive numbers. And he has learned in the preceding chapters how to perform the first two of these operations upon the more general kind of number that we have been considering, that is, upon numbers that may be positive or negative.

We have now to consider the process of multiplication of numbers of this kind. If we were asked to get the product of -5 and 2, we should first have to ask what is meant by the product of a negative and a positive number. This was not explained in arithmetic because there such a problem could not arise. But we can use the definition of the product of two numbers, as given in arithmetic, to enable us to formulate a definition of the product of two numbers of the kind we are here considering that will be a natural and easy extension of the original definition.

The product of 5 and 2 is equal to $5+5$, or 10; and we should accordingly expect this new definition of multiplication to be such that the product of -5 and 2 would equal $-5-5$, or -10. The student can easily verify from the definition of § 44 that this is the case. We should also naturally expect this new definition to be such that the product of two factors be unchanged by the change in the order of the factors, since this is true in arithmetic. For example, $5 \times 2 = 2 \times 5$. This means that we should expect the product of 2 and -5 to be the same as the product of -5 and 2, which is -10.

Moreover we have just said in substance that we should expect this new definition of multiplication to be such that the product 5×2 and -5×2 be the same except for sign. It would be natural then to expect further that the products -5×2 and -5×-2 be the same except for sign. But we have just seen that -5×2 should be -10. Hence we should expect -5×-2 to be 10.

We have made these remarks for the purpose of making the definition of multiplication we are about to give seem a natural extension of the multiplication already familiar to the student. And for this purpose we selected the simplest illustrations possible. We do not mean to imply that in arithmetic multiplication always consists in taking one factor as many times as there are units in the other one, since this has no meaning when the latter is not a positive integer.

44. Definition of Product. — The consideration of these special cases makes it natural to define multiplication in general in the following way:

The product of two numbers of like sign is the product of their absolute values, and the product of two numbers of unlike signs is the negative of the product of their absolute values.

From this definition of multiplication we get at once the following

Rule. — *The product of two numbers of like sign is a positive number and the product of two numbers of unlike signs is a negative number.*

This is called the **Rule of Signs** for multiplication.

45. Signs of Multiplication. — To indicate that one number is to be multiplied by another we sometimes, as in arithmetic, write one of the numbers after the other with the sign × between them. Thus 47×34 indicates that 47 is to be multiplied by 34. Sometimes a dot (·) placed a little above the middle of the line is used. Thus, $47 \cdot 34$. When one or both of the factors are literal it is customary to omit the sign of multiplication between them. Thus $7a$ indicates the product of 7 and a, and bc indicates the product of b and c. (See § 7.)

The product of three numbers is the number obtained by multiplying the product of the first two by the third. Thus, $-7 \times 3 \times 4 = -21 \times 4 = -84$. The product of four or more factors is formed in a similar way.

46. Product of Monomials. — The product of $9a^3b^2$ and $4a^4bc$ is $9a^3b^2 \cdot 4a^4bc = 36a^3a^4b^2bc$. Now it is clear from the meaning of a^3 and a^4 that $a^3a^4 = a^7$ and similarly that $b^2b = b^3$. Hence, $9a^3b^2 \cdot 4a^4bc = 36a^7b^3c$. This example illustrates the following general

Principle. — *The product of two monomials is a monomial in which the numerical coefficient is the product of the numerical coefficients of the factors and in which the exponent of any letter equals the sum of the exponents of that letter in the factors.*

NOTE. — It appears from the definition of an exponent in § 8 that only positive whole numbers can be exponents. But we shall later so modify this definition that other kinds of numbers can be used as exponents. (See Second Course.)

The student should observe that this principle assumes that the factors of a product can be rearranged without affecting the product. In the preceding illustration, for example, we said in effect that $9a^3b^2 \cdot 4a^4bc = 9 \cdot 4a^3a^4b^2bc$. That this assumption is a proper one follows from the definition of multiplication and the fact with which the student is familiar from his study of arithmetic, that the product of two or more positive numbers is the same regardless of the order in which these numbers are taken. If the factors of a product could not be rearranged without affecting the product, algebra would be much more difficult than it is.

EXERCISES

Perform the following indicated multiplications:

1. 7×5. 3. $6 \cdot -4$. 5. $-7 \cdot 15$.
2. $12 \cdot 8$. 4. $15\frac{1}{2} \cdot -7$. 6. $-2\frac{2}{3} \cdot 7\frac{5}{8}$.
7. $-7\frac{5}{8} \cdot 2\frac{2}{3}$. 9. $-3 \cdot -3 \cdot -3 \cdot -3$.
8. $-2 \cdot -10 \cdot -\frac{1}{4}$. 10. $-1 \cdot -17$.

Two or more monomials that contain literal factors are said to be multiplied together when their product is indicated in the simplest possible way.

11. $3 \cdot xyz$. 21. $3\,xyz \cdot 3\,xyz \cdot 3\,xyz$.
12. $14\,a^3 \cdot b^2$. 22. $(3\,xyz)^3$.
13. $10 \cdot a^3xy$. 23. $(-ab^2c^3)^3$.
14. $-10 \cdot -a^3xy$. 24. $(-8\,a^2cmn^3)^4$.
15. $-2\,a^2b^3 \cdot 13\,a^3b^2$. 25. $(-5\,m^2n^4)^3$.
16. $3\,x^2y \cdot -5\,z$. 26. $(-5\,m^2n^4)^4$.
17. $-ab \cdot -8\,ab$. 27. $(x^3y^4z^2)^5$.
18. $4\,a^3bc \cdot -21\,ab^4c^5$. 28. $(4\,m^4n^4)^4$.
19. $25\,m^2n \cdot 3\,n^3x^2y^3$. 29. $(-a)^5$.
20. $3\,xyz \cdot -3\,xyz$. 30. $(7\,ax \cdot -3\,xy)^2$.

31. An automobile goes due east from a certain place for 3 hours at the rate of 15 miles per hour. It then turns around and goes due west for 2 hours at the rate of 20 miles per hour. Where is it at the end of the 5 hours?

32. Explain fully the statement on p. 54 that $a^3 \cdot a^4 = a^7$.

47. **The Degree of a Monomial.** — By the **degree** of a monomial is meant the sum of the exponents of all the literal factors of the monomial.

Thus the degree of $12\,x^2yz^3$ is 6.

In finding the degree of a term the student should not forget that if no exponent is written after a literal factor the exponent 1 is understood.

What are the degrees of the following monomials:

$$9\,abc,\ xyz^4,\ 17\,k^2l^3m^4,\ 4\,x,\ 10\,y^2,\ 5\,a^3b^3c^3\,?$$

The degree of the product of two monomials is evidently the sum of the degrees of the monomials.

When we say that a given monomial is of a certain degree in certain specified letters we mean that the sum of the exponents of these letters in the monomial is equal to this degree. Thus $4\,x^2yz^3$ is of the third degree in x and y, and $5\,a^2b^3m^4n$ is of the fifth degree in a and b.

48. Homogeneous Polynomials. — If all the terms of a polynomial are of the same degree, the polynomial is said to be **homogeneous** and of this degree.

Thus $a^4 + 4\,a^3b + 6\,a^2b^2 + 4\,ab^3 + b^4$ is homogeneous of degree 4.

49. Degree of a Polynomial that is not Homogeneous. — If the polynomial is not homogeneous, the degree of the term of the highest degree is taken as the degree of the polynomial.

Thus the polynomial $4\,x^3y^2 + 3\,xy + 10\,xy^3 - 4\,y^2$ is of degree 5.

50. Arrangement of a Polynomial. — The product of several factors each equal to a is called a **power** of a. The number a is the *first power* of itself.

Thus, the monomial $9\,a^3b^2c$ contains the *third power* of a, the *second power* of b, and the *first power* of c.

A polynomial is said to be *arranged* according to the *descending* powers of a letter if the exponents of this letter do not increase in successive terms from left to right.

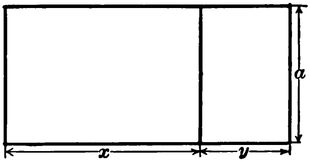
Thus $x^5 - 10\,x^4 + 6\,x - 5$ is arranged according to descending powers of x; also $a^4b^3c - 5\,a^3b^2c^2 + 7\,a^3bc^3 - 10\,a^2bc + 5$ is arranged according to the descending powers of a.

The student should observe that in this case the exponents of a in the successive terms from left to right do not always decrease — the second and third terms have the same power of a; but these exponents do not increase.

When is a polynomial said to be arranged according to the ascending powers of a letter?

It is possible to arrange certain polynomials both according to the descending powers of one letter and the ascending powers of another letter; as for example, $m^5 + 5\,m^4n + 10\,m^3n^2 + 10\,m^2n^3 + 5\,mn^4 + n^5$.

51. Multiplication of Polynomials. — The student is familiar with the fact that the area of a rectangle is equal to the product of its base and altitude. If we have two rectangles with the common altitude a and bases x and y respectively, their combined area is $ax + ay$. But they are together equivalent to a single rectangle with altitude a and base $x + y$. Hence their combined area is also the product of $x + y$ and a; that is, the product of $x + y$ and a is equal to $ax + ay$. This illustrates what is true of the product of any polynomial and a monomial; namely, this product is a polynomial whose terms are obtained by multiplying the terms of the polynomial by the monomial.



For example, the product of $7\,x^2y^3 - 4\,xy - 3\,y^5$ by $-3\,abx^2$ is $7\,x^2y^3 \cdot -3\,abx^2 - 4\,xy \cdot -3\,abx^2 - 3\,y^5 \cdot -3\,abx^2$, or $-21\,abx^4y^3 + 12\,abx^3y + 9\,abx^2y^5$.

Suppose we wish to get the product of $x^2 + 2\,xy + y^2$ and $x + y$.

We can indicate this product thus $(x^2 + 2\,xy + y^2)\,(x + y)$.

If now we look upon the first factor as a monomial and apply the preceding rule, we get

$$(x^2 + 2\,xy + y^2)\,(x + y) = x(x^2 + 2\,xy + y^2) + y\,(x^2 + 2\,xy + y^2)\,;$$

and a second application of the preceding rule shows that this equals

$$x^3 + 2\,x^2y + xy^2 + x^2y + 2\,xy^2 + y^3 = x^3 + 3\,x^2y + 3\,xy^2 + y^3.$$

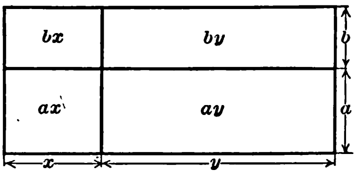
The method used here is of general application and enables us to establish the

Principle. — *The product of two polynomials is a polynomial whose terms are formed by multiplying every term of one polynomial by every term of the other.*

If the product thus obtained is not in its simplest form, it should be simplified. But the work of multiplying can be so arranged as to give the simplified product directly. For example, the product of $4m^3 + 6m^2n - 8mn^2 + 3n^3$, and $-2m^2 + 6mn - n^2$ is obtained as follows:

$$\begin{array}{lrrrrrr}
 & 4m^3 & +\ 6m^2n & -\ 8mn^2 & +\ 3n^3 & & \\
 & -2m^2 & +\ 6mn & -\ n^2 & & & \\
\hline
 & -8m^5 & -\ 12m^4n & +\ 16m^3n^2 & -\ 6m^2n^3 & & \\
 & & +\ 24m^4n & +\ 36m^3n^2 & -\ 48m^2n^3 & +\ 18mn^4 & \\
 & & & -\ 4m^3n^2 & -\ 6m^2n^3 & +\ 8mn^4 & -\ 3n^5 \\
\hline
\textit{Product.} & -8m^5 & +\ 12m^4n & +\ 48m^3n^2 & -\ 60m^2n^3 & +\ 26mn^4 & -\ 3n^5
\end{array}$$

52. Geometrical Interpretation of the Product of Two Binomials. — The simple case of the product of two binomials like $x + y$ and $a + b$ has a natural geometric interpretation. This product is



$$ax + ay + bx + by.$$

If now we consider a rectangle with altitude $a + b$ and base $x + y$, the area, which is this product, is seen by reference to the figure to be made up from four rectangles whose areas are the terms of the product.

53. Checks for Multiplication. — When we form the product of two polynomials by the foregoing rule the numerical value of the product for any set of values of the letters should equal the product of the numerical values of the factors for this set of values of the letters.

For example, consider the following case:

$$\begin{array}{lllllll}
3a & -\ 2b & +\ 4c & & & & \\
\underline{2a} & \underline{+\ 5b} & \underline{+\ 3c} & & & & \\
6a^2 & -\ 4ab & +\ 8ac & & & & \\
 & +\ 15ab & & -\ 10b^2 & +\ 20bc & & \\
 & & +\ 9ac & & -\ 6bc & +\ 12c^2 & \\
\hline
6a^2 & +\ 11ab & +\ 17ac & -\ 10b^2 & +\ 14bc & +\ 12c^2 & = 6+11+17-10+14+12=50.
\end{array}$$

Check for $a = 1$, $b = 1$, $c = 1$.

$$\begin{array}{r}
3-2+4 = 5 \\
2+5+3 = \underline{10} \\
50
\end{array}$$

[*Product*.

The product of two homogeneous polynomials is evidently a homogeneous polynomial whose degree is equal to the sum of the degrees of the two factors. Thus the product of $a^2+2ab+b^2$ and $a^3+3a^2b+3ab^2+b^3$ is a homogeneous polynomial of degree 5. This general fact can be used as a check when homogeneous polynomials are multiplied together. If the product is not homogeneous, we may know that a mistake has been made somewhere.

When we wish to form the product of two polynomials it is generally advisable to arrange them both according to the ascending or the descending powers of a common letter, if there be one.

EXERCISES

In each of the following exercises form the product of the two polynomials and check your work. When the two polynomials are homogeneous, use also the check described in § 53.

1. $x-5,\ 4x.$
2. $4a^2-6b,\ 3ab.$
3. $x^2+2x+1,\ -6x.$
4. $-4m^2+6m-5,\ 6m^3.$
5. $a^4-4a^3-3a^2-2a+1,\ 8abc.$
6. $5abc^2-2ab^2c+4a^2bc,\ -14a^2b^2c^2.$
7. $a+4b,\ a+3b.$
8. $x+y,\ x+y.$
9. $x-y,\ x+y.$
10. $x+5,\ x+1.$
11. $x-3,\ x+2.$
12. $a-4,\ a-3.$

13. $a + 3,\ a - 2.$

14. $m + 4,\ m + 4.$

15. $x - \frac{2}{3},\ x + \frac{1}{2}.$

16. $x - \frac{1}{2},\ x + \frac{1}{2}.$

17. $a^2 + 8\,ab + b^2,\ a + b.$

18. $x^2 + xy + y^2,\ x - y.$

19. $m^2 - my + y^2,\ m + y.$

20. $x^2 + xy + y^2,\ x^2 - xy + y^2.$

21. $am + bn,\ cm + dn.$

22. $a^3bc^2 - 6\,a^2b^2c + 5\,ab^2c^2,\ ab + bc + ca.$

23. $x + y + z,\ x + y + z.$

24. $a^2 + b^2 + c^2 - ab - bc - ca,\ a + b + c.$

25. $2\,x - 4\,x^5 + 7\,x^2 + 3\,x^4 - x^3 - 1,\ 4 - x^2 + 5\,x.$

26. $3\,a^2b + 3\,ab^2 + a^3 + b^3,\ a^2 + b^2 + 2\,ab.$

27. $2 + 6\,y - 3\,y^2 + 12\,y^3,\ 5 - y.$

28. $1 + y + y^2 + y^3 + y^4,\ 1 - y.$

29. $1 - y + y^2 - y^3 + y^4,\ 1 + y.$

30. $x - 2\,y + 3\,z - 4\,w,\ x - 2\,y + 3\,z - 4\,w.$

31. $x^4 - 4\,x^3y + 6\,x^2y^2 - 4\,xy^3 + y^4,\ x - y.$

32. $x + y + z,\ x + y - z.$

33. $a^3b^2 - 8\,a^2b^3 + a^4 + b^4,\ -9\,a^3b^2.$

34. $a + b + x + y,\ a + b - x - y.$

35. $a^2 + ab + b^2,\ a^2 - ab + b^2.$

36. $m^3 + m^2n + mn^2 + n^3,\ m - n.$

37. $x^3 - y^3,\ x^3 + y^3.$

38. $2\,a - 3\,b + 5\,c,\ 2\,a - 3\,b + 5\,c.$

39. $3\,x - 2\,y - 4,\ 3\,x - 2\,y + 5.$

40. $a + b + c + d,\ a + b + c + d.$

41. $a^{2x} + 2\,a^x + 1,\ a^{2x} + a^x + 1.$

42. $a^m + b^m,\ a^m - b^m.$

43. $x^{2m} + x^my^n + y^{2n},\ x^m - y^n.$

44. $x^2(a + b)^2 - x(a + b) + 1,\ x(a + b) + 1.$

45. $3(x + y) + 2(x - y),\ 3(x + y) - 2(x - y).$

46. $(a + b)^3 + 3(a + b)^2 + 3(a + b) + 1,\ (a + b) + 1.$

DIVISION

54. Definition of Division. — When we are asked to divide 12 by 3 we are in reality asked to find a number such that its product by 3 is 12; and we say that this quotient is 4 because we know that $4 \cdot 3 = 12$. Moreover, this is true of division in general; that is, we say that the result of dividing a by b is c, if $cb = a$.

In accordance with this definition, there can be no division by zero, since there is no number c such that $c \cdot 0 = a$, when a is different from zero; and when a is zero, $c \cdot 0 = a$ for all values of c.

55. Rule of Signs for Division. — If a and b are both positive, we know from the rule of signs for multiplication that c must be positive.

If a is positive and b is negative, we know from the same rule that c must be negative.

If a is negative and b is positive, c must be negative.

If both a and b are negative, c must be positive.

Hence the

Rule. — *The quotient of two numbers with like signs is a positive number and the quotient of two numbers with unlike signs is a negative number.*

This is called the **Rule of Signs** for division.

The student should notice the similarity between this rule and the Rule of Signs for multiplication.

The symbols $a \div b$ and $\frac{a}{b}$ represent the result of dividing a by b. The fractional form is the one more frequently used.

HISTORICAL NOTE. — Most of the subject matter of elementary algebra and geometry has been known for thirteen hundred years and much of it for an even longer time. This is doubtless the reason for the common impression that mathematics is not a growing science, like physics and chemistry, for example. The impression is, however, a mistaken one, as is shown by the fact that the last volume of a German book published annually and containing synopses of the articles that have appeared during the year in the various mathematical journals of the world contains 1110 pages.

The oldest mathematical book known is an Egyptian papyrus now in the British Museum. It was written by a priest named Ahmes probably some time before 1700 B.C. and was founded on a still older work. It contains some arithmetic and some algebra (see note on p. 82.)

As one might expect, the symbols used in the early times were different from those with which we are familiar. For example, Ahmes indicated addition by the picture of a bird, or some animal, with its feet turned in the same direction as its head, and subtraction by the same symbol with the feet turned in the opposite direction. Three horizontal arrows were also used to indicate subtraction and the sign $\leqslant$ was used to indicate equality. The signs for addition and subtraction with which we are familiar date back to about 1500 A.D. when a German named Johann Wideman published a book in which they were used. The sign × for multiplication and the sign = for equality are of English origin. The former was first used by William Oughtred in 1631, and the latter by Robert Recorde in 1557. Recorde was professor of Mathematics and Rhetoric at Oxford University. He died in prison in London. The sign ÷ for division was introduced by J. H. Rahn, a Swiss, in 1659.

56. The Quotient of Two Monomials. — Since $a^4a^3 = a^7$, it follows that $\frac{a^7}{a^3} = a^4$; and, in general, since $a^{m-n}a^n = a^m$, it follows that $\frac{a^m}{a^n} = a^{m-n}$, provided that m is greater than n.

Hence the following

Principle. — *The quotient of two monomials is a monomial whose numerical coefficient is the quotient of the numerical coefficient of the dividend and that of the divisor, and in which the exponent of any letter equals its exponent in the dividend minus its exponent in the divisor.*

If a letter has the same exponent in both dividend and divisor, it will not appear in the quotient. The case in which a letter has a greater exponent in the divisor than in the dividend will be considered in Chapter XI.

EXERCISES

Perform the following indicated divisions:

1. $10 \div 2.$
2. $15 \div -5.$
3. $0 \div 4.$
4. $-18 \div -6.$
5. $\frac{-9}{-4}.$
6. $\frac{0}{-5}.$

A monomial that contains one or more literal factors is said to be divided by another monomial when the quotient of the two is indicated in the simplest way possible.

7. $\frac{4a^3b^4c^5}{-21ab^2c^3}.$
8. $\frac{3x^2yz^3}{3x^2yz^3}.$
9. $\frac{25m^2n^3x^3y^3}{-5m^2n}.$
10. $\frac{-14a^3b^2}{14a^3b^2}.$
11. $\frac{a^xb^2}{ab}.$
12. $\frac{26x^3y^3z^4}{13x^3y^3z^4}.$
13. $\frac{12abc^4}{-4}.$
14. $\frac{-7m^7n^4}{-3m^3n^4}.$
15. $\frac{x^2y^5z^7}{x^2y^5z^7}.$
16. $\frac{22a^3b^{10}z^5}{-22a^2b^{10}z^5}.$
17. $\frac{-19m^3n^4}{3mn^2}.$
18. $\frac{-5xyz}{-2x}.$
19. $\frac{13abm}{-4m}.$
20. $\frac{-21a^xb^y}{-3ab}.$
21. $\frac{rst}{rs}.$
22. $\frac{-rst}{rs}.$
23. $\frac{rst}{-rs}.$
24. $-\frac{rst}{rs}.$
25. $\frac{24a^4b^2}{8a^4b^2}.$
26. $\frac{9x^3y^2z^3}{9x^3y^2z^3}.$
27. $\frac{7r^3s^3}{-3rs^3}.$
28. $\frac{a^4b^3c^2}{-5a^4b^3c^2}.$

29. $\dfrac{-3\,xyz}{-2\,xy}.$

30. $\dfrac{a^3b^3c^3}{9\,a^2b^2c^2}.$

31. $\dfrac{-15\,x^2ab^3}{3\,xab}.$

32. $\dfrac{x^my^n}{3\,xy}.$

33. $\dfrac{-4\,a^{x+1}b^{y+2}}{3\,a^xb^y}.$

34. $\dfrac{7\,a^{m+2}b^{m+3}}{5\,a^mb^{m+1}}.$

35. $\dfrac{-20\,abxy}{8\,ax}.$

36. $\dfrac{34\,m^4n^4}{-17\,mn^4}.$

37. $\dfrac{13\,x^3y^3z^4}{26\,x^3y^3z^4}.$

38. $\dfrac{26\,a^3b^2}{26\,a^2b^2}.$

39. $\dfrac{a^4b^3c^2d}{-a^3b^2c}.$

40. $\dfrac{5\,xy^2z^4}{3\,yz}.$

41. $\dfrac{10\,a^2(x+y)^3}{-2\,a(x+y)}.$

42. $\dfrac{4\,x^2y^3(a-b)^4}{-8\,(a-b)^3}.$

43. $\dfrac{7\,mn\,(a^2+b^2)}{m\,(a^2+b^2)}.$

44. $\dfrac{-14\,a^3b^4(x+y+z)^5}{7\,a^2b^2(x+y+z)^3}.$

45. $\dfrac{4\,x(m^2-mn+n^2)^3}{x(m^2-mn+n^2)^2}.$

57. Division of a Polynomial by a Monomial. — It follows at once from the definition of division and the principle governing the multiplication of a polynomial by a monomial that the division of a polynomial by a monomial is governed by the following

Principle. — *The result of dividing a polynomial by a monomial is a polynomial whose terms are the terms of the dividend divided by the divisor.*

For example, if we divide $8\,a^3b^2 + 5\,a^2b^4 - 12\,ab^4$ by $2\,ab^2$, the quotient is $8\,a^3b^2 \div 2\,ab^2 + 5\,a^2b^3 \div 2\,ab^2 - 12\,ab^4 \div 2\,ab^2 = 4\,a^2 + \frac{5}{2}\,ab - 6\,b^2$.

The case in which any literal factor has a greater exponent in the divisor than in any term of the dividend will be considered later.* (Compare Chapter XI.)

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* The division of a polynomial by a polynomial will be explained in Chapter VIII.

EXERCISES

Divide:

1. $x^3 + 2x^2 + x$ by x.
2. $3a^3b + 24a^2b + 42ab$ by $3ab$.
3. $ab^2c + 5abc + 6ac$ by ca.
4. $4x^4 + 12x^3y + 12x^2y^2 + 4xy^3$ by $4x$.
5. $8a^2b - 8ab^2 - 448b^3$ by $-8b$.
6. $62x^2y^2 - 10xy^3 + 56x^3y$ by $2xy$.
7. $6x^5 + 12x^4 - 18x^2 + 5x$ by $-6x$.
8. $5x^2yz - 10xy^2z + 15xyz^2$ by $5xyz$.
9. $4r^2s^2t^2 - 16r^3s^3t^3 + 20r^4s^4t^4$ by $4r^2s^2t^2$.
10. $a^2b^2c - abc^3$ by abc.
11. $-r^3sab^2 + rt^2a^2b^3$ by $-rab^2$.
12. $m^3nx^2y - m^4n^2x^3y^2$ by $-m^2nx^2y$.
13. $ax^2y^3z - abx^3y^2z + ax^2yz^2$ by $-axyz$.
14. $-a^3b^2c + a^4bc^2 - a^2b^3c$ by a^2bc.
15. $6x^2yza^3 - 9xy^2z^2a^2$ by $-3a^2xyz$.
16. $-50b^5c^4 - 25b^4c^3$ by $25b^4c^3$.
17. $20a^4b^5c^3 - 15a^3b^4c^4 + 12a^5b^3c^2$ by $5a^2b^3c$.
18. $20klm - 15k^2l - 12kl^2 + 5km^2$ by $-3k$.
19. $13a^4b^2c^3 - 5a^3bc^4 - 2a^2bc^3$ by $2a^2bc^3$.

Perform the following indicated divisions:

20. $\dfrac{12a^2 - 4ab}{-2a}$.

21. $\dfrac{12mn - 18m^2}{3m}$.

22. $\dfrac{9x^2 - 18x + 9}{9}$.

23. $\dfrac{25r^2s - 35rs^4}{-5rs}$.

24. $\dfrac{3a^3b + 6a^2b^2 + 3ab^3}{3ab}$.

25. $\dfrac{16a^3c^5 - 24a^5c^3}{4a^3c^3}$.

26. $\dfrac{6b^4a - 12b^6a^2 + 18b^8a^4}{-6ab^4}$.

27. $\dfrac{6\,xyz - 12\,x^2y^2z^2 + 9\,x^3y^3z^3 + 15\,x^4y^4z^4}{3\,xyz}.$

28. $\dfrac{5\,x^{a+4} - 3\,x^{a+3} - 7\,x^{a+2}}{-\,x^2}.$

29. $\dfrac{3\,a^5 - 6\,a^{m+4}b + 9\,a^3b^m}{3\,a^3}.$

30. $\dfrac{12\,a^{m+2}b^{n+3} + 8\,a^{m+1}b^{n+4} + 4\,a^mb^n}{4\,a^mb^n}.$

31. $\dfrac{5(x-2) + 3\,a(x-2)}{(x-2)}.$

Consider $x - 2$ as a single number rather than as the difference of the two numbers.

32. $\dfrac{3\,x(a-b) - 2\,y(a-b)}{(a-b)}.$

33. $\dfrac{a^2bc(m+2\,n) + 4\,ab^2c(m+2\,n)}{abc(m+2\,n)}.$

34. $\dfrac{5(x+y)^3 - 10(x+y)^2 + 5(x+y)}{(x+y)}.$

35. $\dfrac{x(a+b+c) + y(a+b+c) + z(a+b+c)}{(a+b+c)}.$

58. Parentheses preceded by Monomial Factors. — From what the student has learned about the multiplication of polynomials he will recognize that from an expression involving parentheses preceded by a monomial the parentheses can be removed by multiplying every term within the parentheses by the monomial. For example,

$$6\,a^2b(3\,ab - 4\,ab^2 + b^3) = 18\,a^3b^2 - 24\,a^3b^3 + 6\,a^2b^4.$$

$$3(x-5) - 6(x^2 - 3\,x + 2) = 3\,x - 15 - 6\,x^2 + 18\,x - 12 = -\,6\,x^2 + 21\,x - 27.$$

EXERCISES

Simplify the following polynomials as much as possible:

1. $2(x-1)-3(x-2)$.
2. $3(x+2)-5$.
3. $2-4(3x+1)$.
4. $x-1-5(2x-3)+4$.
5. $-7(x-2)+5(1-x)$.
6. $8(x-1)-4(2x+3)$.
7. $-4(y+2z)-3(x+y)$.
8. $3a(x-4)-2b(5-x)$.
9. $2(ax^2+bx)-3(cx^2-dx)$.
10. $-(3a-2b+c)+5(-a-b+c)$.
11. $3x-2(x-6)-2x-3(x-7)$.
12. $1-4(x-2)+3-4(x-5)$.
13. $5x-3(2-x)+3-5(x-2)-4x-3(x+1)$.
14. $3(a^2-6a+7)-2(3-a^2+4a)+5$.
15. $a-(b+3)+3(a-4)-(b+8)+6$.
16. $4(x^2-6x+5)-3(x^2+x-1)$.
17. $3(x+1)-4(x-2)+5(x+3)$.
18. $2a-b-a^2b(x+y-2z)$.
19. $6m(3a-2y+z)-5n(4x-3y-6z)$.
20. $a(3mx^2-5ny^2)-b(2nx^2+my^2)$.
21. $-4(by^2-cy+1)+7(4+y^2-y)$.
22. $9(lx^2+mx-n)-7(3mx-2n+lx^2)$.
23. $5(4a-6b+2c)-10(c+2a-3b)$.
24. $[3x-(2y-z)+4-3(y-2z)]-[5x-4\{x-(y-z)\}]$.
25. $x-(y+z)+3\{x-(z-y)\}+5\{x-2(y+z)\}$.
26. $(x^2-y^2-z^2)+2(x^2-\overline{y^2-z^2})+3(x^2-y^2)$.
27. $[a-3(b+c)+1]-[a+b-c-3(a-2)]$.
28. $3\{x-(y+z)+3(y-z)\}-4\{-x-y+2(z-x)\}$.
29. $[5a+2(3b-2c)]-[-3(3b-5a)+4a]$.
30. $-\{5x[2x-(4x+3b)+6x]-3x(b-2)\}$.

59. A knowledge of the principles of multiplication and division explained in this chapter will enable the student to solve the following exercises and problems which are very similar to those of § 38.

EXERCISES AND PROBLEMS

Find the numbers that answer the following descriptions. Be sure to have a good reason for every step you take.

1. $2(x-3)=10.$
2. $3+3(2-m)=0.$
3. $5(r-1)+2(3-r)=1.$
4. $-2(4-y)+3(1-y)=4.$
5. $z-1+2(5-z)=0.$
6. $2(3-y)+7(y+6)=0.$
7. $x-2+(x+4)=0.$
8. $l+2l+12=5.$
9. $2z-4z+13-10z=25.$
10. $3t+6=0.$
11. $2(x-1)+3(5-x)=0.$
12. $x+7+2(-3-x)=0.$
13. $x-5-3(x-9)=0.$
14. $x+8-(10-x)=0.$
15. $2(x+6)=-14.$
16. $2(s-1)+3(s-2)+4(s-3)=-20.$
17. $5(2y+7)+4(3y-6)+2y-3=4.$
18. $2(x-1)-3(x-2)+4(x-3)=20.$
19. $7(z-5)+45+3z=0.$
20. $3(5r-1)-6r-3(r-4)=4.$
21. $1-4(2x-7)-3(x-5)=0.$
22. $7(x+4)+10=80.$
23. $2n-2(3n-4)-5(2-4n)=6.$
24. $4(x-1)-(x+5)=-3.$
25. $12+3(x-3)-(x-2)=0.$
26. $3(x+1)+2(x+2)-(x+3)=4.$
27. $7(r-6)+10r-5(r-1)=8.$
28. $-3(5x+3)+4(2x-3)=-70.$

29. $6y + 3(2 - y) - 4(y - 6) = 10.$

30. $4(x + 5) + x - 8 = -13.$

31. $10x - 4 - (3x + 7) = -11.$

32. $2(t - 1) - 3(t + 1) = -3.$

33. $2(8x + 1) - 4(3x - 5) = 58.$

34. $3(x - 3) + 3x - 5(x - 1) = 3.$

35. $6(y - 1) - 4(y + 1) = 6.$

36. $7(2y - 4) - 4(3y - 3) = 0.$

37. $10(2 - 6x) - 4(2x - 10) = -8.$

38. $4(9 - x) - (1 - 9x) = 0.$

39. $4(x - 1) - (1 - 9x) = 34.$

40. $2(r + 3) - 3(r + 2) = 1.$

In each of the Exercises 41–50 draw an axis and locate the points mentioned.

The points A and B have the abscissæ -10 and 8 respectively.

41. What is the abscissa of P if $AP + 2\,PB = 0$?

42. What is the abscissa of Q if $2\,AQ + 3\,QB = 0$?

43. What is the abscissa of R if $4\,AR + 5\,RB = 0$?

44. What is the abscissa of X if $3\,XA + BX = 0$?

45. What is the abscissa of Y if $AY + 3\,YB = 0$?

The points A, B, and C have the abscissæ 12, 0, and 20 respectively.

46. Where must the point X be in order that $AX + 2\,CX = AB$?

47. Where must the point Y be in order that $BY + 2\,YC = 3\,BC$?

48. Where must the point Z be in order that $3\,ZB = BA + 2\,AC$?

49. Where must the point P be in order that $2\,BP = 2\,BA + CA$?

50. Where must the point Q be in order that $BQ + AQ + CQ = 0$?

Parcel Post Rates. — The parcel post rates to points in the first five zones are determined by the formula

$$r = 5 + 3\,(p-1) + p\,(z-1).$$

where r is the rate in cents, p the number of pounds, and z the number of the zone. If p is a fraction the next greater integer must be substituted for it in the formula.

If the parcel is insured the total rate to points within the first five zones is given by the formula

$$r = 15 + 3\,(p-1) + p\,(z-1).$$

51. What is the cost of sending 8 pounds to the third zone?

52. How many pounds can be sent to the fifth zone for 65 cents?

53. How many pounds can be sent to the fourth zone for 56 cents?

54. How many pounds can be sent to the fourth zone for 54 cents, including the cost of insurance?

55. How many pounds can be sent to the second zone for 56 cents, including the cost of insurance?

56. If a system of pensions is based on the formula

$$\frac{A}{100}\,(b + 10) + 300 = P,$$

where A = active pay, P = annual pension, and b = number of years of service, what must be the salary of a man after 40 years of service in order that he may retire on an annual pension of $2000?

57. A man with a salary of $3000 a year is retired under the system described in Ex. 56 on an annual pension of $1500. How many years of service did he have to his credit?

REVIEW OF CHAPTERS I–V

1. Define monomials and polynomials, and give an example of each.

2. What are similar monomials? Give an example of similar monomials and an example of dissimilar monomials.

3. Define and illustrate the terms exponent, coefficient, literal coefficient, and literal factor.

4. What is meant by the algebraic sum of two numbers?

5. What is meant by the absolute value of a number? Name two unequal numbers that have the same absolute value.

6. Describe the advantage in using negative numbers. Cite some circumstances not given in the text in which positive and negative numbers can be used to advantage.

7. Define a segment of a line. What is the difference between the segment AB and the segment BA?

8. Explain what is meant by the abscissa of a point.

9. What is a binomial? Give an illustration.

10. What is a trinomial? Give an illustration.

11. State the *Rule of Signs* for multiplication.

12. State the *Rule of Signs* for division.

13. Why were there no similar rules in Arithmetic?

14. What is meant by the fifth power of a number?

15. What is the product of a^5 and a^3? Explain fully.

16. What is an algebraic expression?

17. Give one or more illustrations from ordinary affairs of the algebraic principle that the effect of subtracting a negative number from any number is the same as that of adding the corresponding positive number to this number.

18. Describe a check for detecting errors in addition and explain the principle on which this check rests.

19. Describe a check for detecting errors in subtraction and explain the principle on which this check rests.

20. Describe a check for detecting errors in multiplication and explain the principle on which this check rests.

21. Describe a check for detecting errors in division and explain the principle on which this check rests.

22. Explain the use of parentheses.

23. State and explain the rule for removing parentheses.

24. State and explain the rule for inserting parentheses.

25. Give a geometrical illustration similar to the one given in § 51 of the fact that $a(x+y+z)=ax+ay+az$.

26. Give a similar illustration for the product $(a+b+c)(x+y+z)$.

27. Give a similar illustration for the product $(a-b)(x+y)$.

28. Define a homogeneous polynomial. What can you say about the product of homogeneous polynomials?

29. What is meant by the degree of a polynomial?

30. In § 17 it was stated that at times the term polynomial would be used to include monomials. What is the advantage of doing this?

REVIEW EXERCISES ON CHAPTERS I–V

Simplify the following as much as possible:

1. $3x^2+5x^2+10x^2+6x^2$.

2. $8ab-3ab+5a\cdot 3b-20ab$.

3. $7r^2s-5rs^2+3r^3-8s^3-2rs^2+10r^2s-20s^3$.

4. $(2x+3y-6z)(-x-5y+4z)$.

5. $(2x+3y-6z)+(-x-5y+4z)$.

6. $(2x+3y-6z)-(-x-5y+4z)$.

7. $(8a^2bc^3+5ab^2c^2-3a^3b^3c)\div abc$.

8. $(8\,a^2bc^3 + 5\,ab^2c^2 - 3\,a^3b^3c)abc.$

9. $(8\,a^2bc^3 + 5\,ab^2c^2 - 3\,a^3b^3c) + abc.$

10. $(8\,a^2bc^3 + 5\,ab^2c^2 - 3\,a^3b^3c) - abc.$

11. $37\,x^2yz^3 - 38\,x^2yz^3.$

12. $(-6\,a^2b^2c^2)^4,\ (\tfrac{4}{3}\,x^2y^3z)^3,\ 3(l^3m^2)(l^2m^3)^3.$

13. $\dfrac{xy^3z^2}{y^3z^2},\ \dfrac{5\,ab^3}{-\frac{1}{2}\,ab^2}.$

14. $a+b+c+d+2\,a-b+3\,c+16\,d+a+2\,b+3\,b-10\,d.$

15. $2(r+s)^2-3(r-s)^2+2(r+s)(r-s).$

16. From $6\,x^2y-3\,y^3+4\,x^3+xy^2$ subtract $x^3+y^3+3\,xy^2+3\,x^2y.$

17. Add $4\,ab^2+3\,a^2b-2\,ab,\ 5\,a^2b+7\,ab^2-9\,ab,$ and $11\,ab.$

18. Add $12\,x^2yz-2\,xy^2z-3\,xyz^2,\ 15\,x^2yz,$
and $-27\,x^2yz+3\,xy^2z+4\,xyz^2.$

19. Add $3\,b+4,\ -10\,b-6,\ 8-5\,b,$ and $-7-b.$

20. Add $3\,a^2x-7\,ax^2+6\,x^3,\ a^3-x^3,\ ax(x+a),$ and $a^3-y^3.$

21. From $3\,a^2x-7\,ax^2+6\,x^3$ subtract $a^3-x^3.$

22. From $3\,b+4$ subtract $-(3\,b+4).$

23. From 0 subtract 0.

24. $0(x^3+4x^2y-6\,xy^2+y^3)-0(3\,x^3-6\,y^3).$

25. $(a+2)^2+(a-1)^2+(a+1)^2-3\,a(a+2)-60\,a-130.$

26. $3(4\,xy-3\,y^2)(2\,x+y)+5(x+y)(x^2+xy+y^2).$

27. $(13\,x^4y-16\,x^3y^4-2\,x^3y^2)\div x^3y.$

28. $6(x-2)+48.$

29. $3[(2\,x+7)-(3\,x+7)]-[-5\,x+2-3(x-1)].$

30. $8-2[5-3\{6+4(6-1)+5\}-10].$

31. $[a+2(b+3)][a-5(b+c)].$

32. $7\,a-3\,b-\{4\,a-[6\,a+(8\,b-10\,a)-(8\,b-a)]\}.$

33. $3-(2\,x+7\,y-2\,z)+4\,y-3(-2\,x-5\,y+3\,z).$

34. Find the number described by the equation

$$4(x-1)+3-10x=5.$$

35. The abscissa of P is 20 and that of Q is 9. What are the abscissæ of these points with respect to the point halfway between them as origin?

36. The abscissa of A is 14 and that of B is -5. What is the abscissa of the point X such that $BX-XA=0$?

37. If $BY-2AY=0$, what is the abscissa of Y?

38. What is the abscissa of the point R if $2PR-3QR=0$, and the abscissæ of P and Q are 7 and 20 respectively?

39. What is the abscissa of the point R if $2PR-3RQ=0$, and P and Q are the same as in the preceding problem?

40. When a certain number is subtracted from 10 the result is 18 less than three times the number. Find the number.

41. Find the number that answers the description

$$9x+4-3x-5-5x=0.$$

42. Chicago and Washington are respectively 87° 37′ and 77° 4′ west of Greenwich. What is the longitude of Chicago measured from Washington?

43. What is the longitude of Washington measured from Chicago?

44. Edinburgh is 3° 11′ west of Greenwich, and Calais, France, is 1° 55′ east. What is the longitude of Edinburgh measured from Calais?

45. The abscissæ of the points A, B, C, and D are 2, 0, -8, and 1 respectively. Where is the point whose abscissa is the average of these?

46. What is the abscissa of the point halfway between A and C?

47. Find the sum of the segments AB, BC, CD, and DA.

48. What are the segments CA, CB, and CD?

49. What is the time at Berlin when it is noon at Rome? It is 49 minutes and 28 seconds after five in Berlin and 45 minutes and 49 seconds after five in Rome when it is noon in New York.

50. What is the time at Rome when it is noon at Berlin?

51. Find the sum of $13x^4y - 16x^3y^4 - 2x^3y^2$ and $x^4y + 4x^3y^2 - 6x^2y^3 + xy^4$.

52. Subtract the second polynomial of Ex. 51 from the first one.

53. Find the product of the polynomials of Ex. 51.

54. It is 5 minutes and 27 seconds after eleven in Chicago when it is noon in New York. What is the time in New York when it is noon in Chicago?

Simplify the following expressions:

55. $\dfrac{7a^2xy^3}{-3ay^2}$.

56. $\dfrac{-12a^xb^yc^{z+2}}{-3a^xc^{z+1}}$.

57. $\dfrac{5(x+y)^3(a-3b)^4}{5(x+y)^3(a-3b)}$.

58. $\dfrac{0}{-7x^3y^2z^4}$.

59. $\dfrac{(a+b)^3+3(a+b)^2+3(a+b)}{(a+b)}$.

60. $(x-y)(x^2+xy+y^2)-(x+y)(x^2-xy+y^2)$.

61. $(a^2+ab+b^2)(a^2-ab+b^2)(a-b)(a+b)$.

62. $(x^3-3x^2+3x-1)(x^3+3x^2+3x+1)$.

63. $0(a^3b^2-3a^2b^3+b^5)+6a^5$.

64. $\dfrac{2a^3b^2-2ab^4}{-2ab^2}$.

65. $a+x+a+2x+a+3x-3a-6x$.

66. $(x+y)a+(x+2y)b+(3x-4y)a-(3x+2y)b$.

67. $(3x^{2m}y^{2n}-5x^{m+4}y^{n+2}+2x^{m+1}y^{n+2}) \div x^my^n$.

68. $(a^{2m}+a^mb^n+b^{2n})(a^{2m}-a^mb^n+b^{2n})$.

69. $(x^{2a}+2x^ay^b+y^{2b})(x^{2a}-2x^ay^b+y^{2b})$.

70. $(a^{3m+n}+b^{3n+m})(a^{3m+n}-b^{3n+m})$.

71. What is the average longitude of Calais, Edinburgh, Washington, and Chicago? (See Exs. 42 and 44.)

72. When it is noon in New York, what is the time at a place whose longitude is midway between that of Berlin and that of Chicago? (See Exs. 49 and 54.)

73. The abscissæ of *A* and *B* are 8 and 11 respectively and the segment *AB* is divided into 3 equal parts. What are the abscissæ of the points of division?

74. What is the abscissa of the point to the right of *B* and twice as far from *A* as from *B*?

75. What is the abscissa of a point to the left of *A* and four times as far from *B* as from *A*?

76. What number answers the description

$$2(3-x)+4(x-6)=2?$$

77. What number answers the description

$$8-x-(2x-8)=0?$$

Identify the numbers that answer the following descriptions:

78. $x+2x-4x=0$.

79. $2(x-3)+3(1-x)=10$.

80. $2(3-x)+3(x-1)=-10$.

81. $7x-2+6x-10(x-6)=12$.

82. $x-10+3(6-x)=12$.

83. $3(x-5)-8(2-x)-4(1+x)=0$.

84. $4x+9-3x+8+2x=32$.

85. $6(x+5)-7(3+x)-9=0$.

86. $10(x-2)+20+10(4+x)=0$.

87. $3x+5=0$.

88. $7x-4=7a+3$.

89. $9(x+5)=0$.

90. $9(x-5)=0$.

91. Where is the point whose abscissa is the algebraic sum of the abscissæ of the points of Ex. 45?

92. Where is the point whose abscissa is the sum of the absolute values of the abscissæ of the points of Ex. 45?

93. If the abscissæ of A and B are a and b respectively, what is the abscissa of the point midway between them?

94. Two sides of a triangle are equal and the third side is 2 inches longer than these two sides together. What is the perimeter of the triangle if the equal sides are each x inches long?

95. A rectangle is 3 inches longer than it is wide and is x inches wide. What is the distance around it?

96. The sum of two numbers is s and one of them is a. What is the other one?

97. The difference of two numbers is d and the smaller one is s. What is the other one?

98. Find the product of $(a+b)+x$ and $(a+b)-x$.

99. Find the sum of the two polynomials of Ex. 98.

100. Subtract the second polynomial of Ex. 98 from the first one.

CHAPTER VI

LINEAR EQUATIONS

60. In finding the answers to some of the questions that have preceded, the student has had occasion to look for numbers that have been described by means of equations. But nothing has yet been said about a general method for finding numbers thus described. We shall now consider a method that will apply to certain descriptions.

61. **Root of an Equation.** — The process of finding a number that is described by means of an equation is called the process of **solving** the equation. A number answering the description is called a **root** of the equation, and is said to **satisfy** the equation.

62. **Members of an Equation.** — That part of an equation that is on the left of the sign of equality is called the **left**, or **first member** of the equation; the part on the right is called the **right**, or **second member**.

63. **Linear Equations.** — If the members of an equation are either known numbers or polynomials of the first degree in the letter representing the unknown number, the equation is said to be of the **first degree**, or **linear**. (See § 78.)

For example, $2x + 5 = 3$, $2 = 6x - 5$, $12x + 1 = 8 - 3x$ are linear equations.

In this chapter we shall confine our attention to linear equations.

Example 1. — Suppose we wish to find the number answering the description

$$x + 3 = 9 - 2x. \qquad (1)$$

The description says that when $2x$ is subtracted from 9, the difference is $x + 3$. Hence if we add $2x$ to $x + 3$, the sum will be 9. (See the definition of subtraction in § 31.) If we state this result in the form of an equation, we have

$$x+3+2x=9, \qquad (2)$$

or

$$3x+3=9. \qquad (3)$$

This last statement says that if we add 3 to $3x$, the sum will be 9. But this is only another way of saying that if we subtract 3 from 9, the remainder will be $3x$; that is,

$$3x=9-3, \qquad (4)$$

or

$$3x=6. \qquad (5)$$

Now this statement says that 3 times a certain number is 6. Hence the number in question must be $\frac{6}{3}$ (see the definition of division, § 54), and we have

$$x=2. \qquad (6)$$

This means that the number described by the original equation is 2. The student should ascertain by trial whether the number 2 does answer the original description or not.

Equation (2) was obtained from (1) by adding $2x$ to each member; equation (4) was obtained from (3) by subtracting 3 from each member; and (6) was obtained from (5) by dividing each member by 3.

Example 2.—Suppose we wish to find the number answering the description

$$3(x-5)+\tfrac{3}{2}x-\tfrac{4}{3}=x-\tfrac{7}{3}. \qquad (1)$$

We observe that 6 is the least common multiple of the denominators of the numerical coefficients, and that therefore the product of each member of the equation by 6 will contain only integral numerical coefficients.

Now since the description says that the number represented by the left member is the same as the number represented by the right member, six times the left member must equal six times the right member. That is,

$$18(x-5)+9x-8=6x-14, \qquad (2)$$

or

$$18x-90+9x-8=6x-14, \qquad (3)$$

or

$$27x-98=6x-14. \qquad (4)$$

This statement says that $6x-14$ is the result of subtracting 98 from $27x$. This is only another way of saying that

$$27x=6x-14+98, \qquad (5)$$

or

$$27x=6x+84. \qquad (6)$$

This last statement says that $27x$ is the result of adding $6x$ to 84, and this is only another way of saying

$$27x - 6x = 84, \tag{7}$$

or

$$21x = 84. \tag{8}$$

If now 21 times a number is 84, the number must be $\frac{84}{21}$, or 4.

Hence,

$$x = 4. \tag{9}$$

That is, 4 is the number described by the original equation.

Equation (2) was obtained from (1) by multiplying each member by 6; (5) was obtained from (4) by adding 98 to each member; (7) was obtained from (6) by subtracting $6x$ from each member; and (9) was obtained from (8) by dividing each member by 21.

64. These examples illustrate the following general

Principle. — *The number described by a given equation is the same as the number described by a new equation formed from the given one by any one of the following processes:*

1. *The addition of the same number to each member.*
2. *The subtraction of the same number from each member.*
3. *The multiplication of each member by the same number.*
4. *The division of each member by the same number.*

From this principle it is clear that when we are looking for the number described by an equation, we can transpose any term from one member of the equation to the other, provided we change the sign of the term, and then look for the number described by the new equation.

NOTE. — The example given in § 128 illustrates certain restrictions to which statement 3 is subject. A somewhat detailed discussion of the restrictions to which statements 3 and 4 are subject is given in the Second Course.

65. In solving a linear equation the student should observe the following

Rule. — 1. *If any of the numerical coefficients are fractions, multiply both members of the equation by the least common multiple of the denominators of these coefficients.*

2. *Remove any parentheses that may occur in either member of the resulting equation.*

3. *Transpose all the terms that involve the unknown number to the left member, and all terms that do not involve the unknown number to the right member.*

4. *Simplify each member of the resulting equation by combining similar terms.*

5. *Divide both members of the final equation by the coefficient of the unknown number.*

The final equation gives a value of the unknown number, and since all the equations of this series describe the same number, the number given by the last equation must be the number described by the first equation.

The application of the first part of this rule produces an equation with integral coefficients; that is, it *clears the equation of fractions.*

It is not necessary that the foregoing operations be performed in the order given. It will sometimes be desirable to perform them in a different order. The student should use his judgment as to the order to be followed in each case.

In solving an equation, the student should avoid following this rule in a mechanical way. In each case he should have clearly in mind at the beginning what it is he wishes to accomplish, and he should have a definite reason for each step he takes.*

66. Checking. — If we substitute the root of an equation for the unknown number in the equation, we should get the same numerical value for each member of the equation.

Thus in the second example worked above, if we put 4 for x, we get $3(4-5)+\frac{3}{2}\cdot 4-\frac{4}{3}$ and $4-\frac{7}{3}$ for the respective values of the two members of the equation. But these two expressions are evidently equal. This proves that 4 is a number described by the equation; that is, that 4 is a root of the equation.

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* We shall later (Chapter XV) consider equations that must be dealt with differently.

67. Identities and Equations of Conditions. — Certain equations, as, for example, $(x+1)^2 = x^2 + 2x + 1$, give descriptions that apply to all numbers. These are called **identities**. An equation that gives a description which does not apply to all numbers is called an **equation of condition**.

The symbol $\equiv$ is sometimes used in place of the equality sign to indicate an identity. Thus $(x+1)^2 \equiv x^2 + 2x + 1$.

Historical Note. — The Egyptians away back in the time of Ahmes (see p. 62) knew how to solve simple linear equations in one unknown. Ahmes tells, for example, how to solve the equation

$$\frac{2x}{3} + \frac{x}{2} + \frac{x}{7} + x = 33.$$

But he did not express the equation in this way. He represented the unknown by the word *heap* instead of by the letter x and his statement was "Heap, its $\frac{2}{3}$, its $\frac{1}{2}$, its $\frac{1}{7}$, its whole, gives 33."

The use of the letter x to represent the unknown is due to Descartes (see p. 110).

The Chinese wrote the positive coefficients in an equation in black and the negative coefficients in red.

EXERCISES

Find the number described by each of the following equations, and check your results:

1. $x + 1 = \frac{4}{5}$.
2. $2x + 18 = 34$.
3. $2y - 5 = y - 4$.
4. $6x - 2 = 8x + 8$.
5. $3(x - 3) + 5x = 2x - 6$.
6. $4 - 6z = 5 - 2z$.
7. $2x - 5 + 6x - 2x - 4 = 39$.
8. $x + x - 1 + x - 2 = 15$.
9. $4z - 5 + 2z = 3z + 4$.
10. $5(x - 2) = 3(2 - x)$.
11. $4x + 8(5 - x) = 28$.
12. $3(2x - 5) + 3 = 4x - 12 + 2x$.
13. $5(3x - 1) - 3 = x - 4(1 + 2x) + 40$.

14. $2(x-5)=3(8-x)$.

15. $6-3z+5(z-2)=3(1-z+13)$.

16. $9(2-6x)=20-4(x-5)$.

17. $6x-5-2(3x-2)=-1$.

18. $2(x+8)=-4(x+5)$.

19. $\frac{5}{2}x+\frac{1}{4}x-\frac{1}{4}=\frac{5}{4}$.

20. $2-4l+\frac{7}{2}=3l+\frac{5}{6}$.

21. $5+\frac{8}{5}(2k-5)=\frac{1}{2}k+3(3+k)$.

22. $3(z-5)+\frac{3}{2}z-\frac{4}{3}=z-\frac{7}{3}$.

23. $4x+\frac{2}{5}(3+x)=6+5x$.

24. $\frac{3}{2}(x+5)-\frac{1}{3}(x-2)=7$.

25. $2y=14-\frac{2}{5}(y-1)$.

26. $\frac{1}{3}(x-10)=\frac{1}{2}(4-4x)+11$.

27. $\frac{1}{2}x+4=x+\frac{2}{3}x-10$.

28. $-\frac{1}{4}x+\frac{3}{5}x+3=\frac{1}{3}x$.

29. $2(9y+1)=2-8(y-1)$.

30. $5(x+2)+4(x+3)+5=0$.

31. $\frac{1}{3}y+3=\frac{2}{5}y$.

32. $3(x-2)+8-4x-(x+2)=-8$.

33. $3-7(2x+3)+6(3x-8)=2(x-5)$.

34. $\frac{3}{5}x-\frac{6}{5}+4(x-1)=4$.

35. $\frac{1}{2}(x+4)-\frac{3}{2}x=\frac{2}{3}(x+5)+2$.

36. What must be the value of x when $xy-2x-3y+1=0$ and $y=3$?

37. What must be the value of x when $xy-3x-4y+1=0$ and $y=2$?

68. Ordinarily the description we have of a number is not expressed in the form of an equation, but in ordinary non-technical terms. In such a case the first step toward finding the number answering the description is to translate the description into the form of an equation, using some letter to represent the number described. Any letter may be used for this purpose, but x is the one commonly used. After the description has been put into the form of an equation, we can proceed toward the solution in the way described in the preceding section.

In general the most difficult part of a problem of this kind is the statement of the conditions of the problem in the form of an equation. Accordingly the following exercises are given in order that the student by working them may acquire facility in this kind of translation.

EXERCISES

Translate the statements of Exercises 1–9 into equations, using some letter to represent the number described:

1. A number increased by 2 is equal to 9.

2. A number diminished by 4 is equal to 6.

3. Four times a number diminished by 3 equals 17.

4. One fifth of a number plus 18 equals four times the number.

5. Two thirds of a number is 2 more than half the number.

6. When 5 is subtracted from a number and the difference multiplied by 2, the product is the same as that obtained by subtracting the number from 10 and multiplying the difference by 3.

7. The excess of a number over -8 is equal to the excess of 9 over the number.

8. Twice the excess of a number over 6 is equal to three times its excess over 12.

9. A number is as much greater than 5 as it is less than 18.

10. The difference between two numbers is 12 and the greater one is a. What is the smaller one?

11. The difference of two numbers is x and the greater is 17. What is the smaller one?

12. The difference between two numbers is 8 and the smaller one is b. What is the larger one?

13. The sum of two numbers is -14 and one of them is x. What is the other one?

14. The sum of two numbers is a, and one of them is 19. What is the other one?

15. The sum of x and another number is 20. What is this other number?

16. The difference between two numbers is b and the greater one of them is a. What is the other one?

17. The difference between two numbers is d and the smaller one is y. What is the greater one?

18. If A is x years old, how old was he 5 years ago? 18 months ago? How old will he be in 12 years?

19. If A is y years old and B is 2 years more than three times as old, how old will each be in 10 years?

20. If A is x years old and B is twice as old, how old will B be in 8 years?

21. A is 35 and in x years B will be twice as old as A. How old will B be in x years?

22. A is 25 and B 30. State that x years ago B was twice as old as A.

23. Two brothers are 6 and 2 respectively. State that in x years the older one will be twice as old as the younger one.

24. What is the sum of a certain integer x and two next greater integers?

25. If $2x + 1$ is an odd number, what is the preceding odd number?

26. What is the sum of the three consecutive odd numbers of which $2x + 1$ is the middle one?

27. What is the sum of four consecutive even numbers of which $2n$ is the smallest one?

28. What is the nth positive even integer?

29. What is the nth positive odd integer?

30. Translate equations 1–35 in the preceding article into ordinary language, calling the letter that appears in the equation "a number."

APPLICATIONS TO VARIOUS PROBLEMS

69. It was said in the introductory chapter that in the study of algebra we should find a systematic means for discovering the information contained in certain kinds of statements, and in corroboration of this we shall apply here what we have learned so far of algebra to the solution of certain important problems.

If a problem has a solution, this solution must be contained in some way in the statement of the problem. And therefore to solve a problem is to discover certain information contained in the statement of it.

In all the problems given here the information to be obtained is the value of a number. In solving them, therefore, the student should proceed in the following way:

1. *Represent the number to be determined by some letter, as x. It is sometimes better to let the letter represent a number closely related to the number to be determined rather than this number itself.*
2. *Translate the statement of the problem into an equation.*
3. *Solve the equation.*

When the number sought has been determined, it should be tested to see if it satisfies the conditions of the problem. It is not sufficient to see if it satisfies the equation obtained, since there may have been a mistake in getting the equation.

PROBLEMS INVOLVING NUMBER RELATIONS

70. Illustrative Problem. — The greater of two numbers is five times the smaller one and the difference of the two is 24. What is the greater number?

Let $x =$ the smaller number.

Then $5x =$ the greater number.

By the conditions of the problem,

$$5x - x = 24.$$

Hence $4x = 24,$

and $x = 6.$

The greater number, which is $5x$, is therefore 30.

Since the greater number is asked for, it would have been natural to let x equal the greater number. But this would have led to an equation with fractions, which can be avoided by letting x equal the smaller number.

It is easy to verify that these numbers satisfy the conditions of the problem.

1. The sum of three consecutive integers is 75. What is the smallest of the integers?

Hint. — Let $x =$ the smallest of the integers. Then $x + 1$ and $x + 2$ represent the others. The sum of all three is $x + (x + 1) + (x + 2)$, and the problem states that this sum is 75.

2. The sum of three consecutive odd numbers is 21. What is the middle one?

3. The sum of three consecutive even numbers is 54. What is the middle one?

4. The sum of four consecutive even numbers is 92. What is the smallest one?

5. The sum of two numbers is 24 and their difference is 6. What are the numbers?

6. Divide 425 into two parts, one of which is four times the other.

7. Divide 150 into two parts, the greater of which is 2 less than three times the smaller.

It is not necessary that the known quantities in a problem be given numerical values. They may be represented by letters.

8. The sum of two numbers is s and their difference is d. What are the numbers?

Let $x =$ the larger of the two numbers.
Then $x - d =$ the smaller one
and $x + x - d = s,$

$$2x = s + d,$$

$$x = \tfrac{1}{2}(s + d), \text{ the larger number.}$$

$$x - d = \tfrac{1}{2}(s + d) - d = \tfrac{1}{2}s + \tfrac{1}{2}d - d = \tfrac{1}{2}(s - d), \text{ the smaller number.}$$

State this result in words.

The advantage of thus representing the known numbers of a problem by letters is that it enables us to formulate a rule in accordance with which every problem of the same kind can be solved.

9. The sum of two numbers is 10 and their difference is 4. What are the numbers?

Solve this problem by making use of the results of the preceding problem.

10. A newspaper account states that in the senate of a certain state legislature 27 members belong to one political party and that they form a majority of 8. Is it possible to determine from this the total membership of the senate?

11. In a legislative body of 48 members, how many must belong to one political party in order that that party may have a majority of 4?

12. Find the number that is as much greater than 10 as it is less than 76.

13. The excess of a certain number over 22 is equal to twice its excess over 30. What is the number?

14. The excess of 18 over a certain number is three times the excess of -2 over the same number. What is the number?

15. A committee of 20, composed of Democrats and Republicans, is to be appointed, and is to contain 50 per cent more Democrats than Republicans. How many of each party will there be on the committee?

16. A is 40 years old and B is 10. In how many years will A be just twice as old as B? Will A ever be five times as old as B?

17. A, who is 41, in trying to recall the age of his friend B, remembers that five years ago he was two thirds as old as B. How old is B?

18. One of two brothers is 9 and the other is 2. In how many years will the younger one be half as old as his brother?

19. What number must be subtracted from each of the numbers 60 and 50 in order that the first difference shall be twice the second one?

20. A is now 50 and B is 44. How long ago was A three times as old as B?

21. A man is three times as old as his son, but in 15 years he will be only twice as old. How old is the son?

22. In the sixteenth century Joseph Scaliger proposed that historical dates be reckoned from a time 4713 B.C. The system of chronology in which this is done is called the *Julian System.* What date in the Christian era was one fourth its date in the Julian System?

PROBLEMS INVOLVING MOTION WITH UNIFORM VELOCITY

71. Formula for Motion with Uniform Velocity. — If a body moves 3 feet every second, in 5 seconds it will move $15 (= 5 \cdot 3)$ feet; and, in general, if it moves v feet every second, in t seconds it will move tv feet. The number of feet the body passes over in a second is called its *velocity* per second. If s stands for the distance, or space, passed over in t seconds, we have the formula

$$s = vt.$$

We have used these letters because they are the initials of the words *space, velocity,* and *time* respectively.

This formula applies only to the cases in which the body passes over the same number of feet every second — that is, to cases of *uniform* velocity. It does not apply to the case of a falling body, for instance, since a falling body goes farther every second of its fall than it did the preceding second, as we shall see later. (Problem 10, p. 253.)

The relation connecting s, v, and t can be expressed in two other ways; namely,

$$v = \frac{s}{t}, \text{ and } t = \frac{s}{v}.$$

The first formula says that in the case of uniform velocity the space passed over by a body in a given time is equal to its velocity multiplied by the time. The second says that the velocity is equal to the space divided by the time, and the third says that the time is equal to the space divided by the velocity.

Sound travels in air with a velocity of 1125 feet a second and light travels with such a velocity that the flash accompanying the report of a gun can be seen practically simultaneously with the explosion.

1. A man on Riverside Drive, New York City, heard a salute from a warship in the Hudson River 2 seconds after he saw the flash. How far away was the ship? If he had been a mile from the ship, when would he have heard the report?

2. A man heard a clap of thunder $1\frac{1}{2}$ seconds after he saw the accompanying lightning. How far away was the thunder?

3. At what uniform rate must a boy walk in order to cover 7 miles in 2 hours?

4. A marksman heard the sound of his bullet on the target 5 seconds after he fired. How far away is the target? Take the average velocity of the bullet to be 1290 feet a second. For the velocity of sound, see the explanation preceding Problem 1.

5. A is standing between B and a target and is 50 feet nearer the target. Six seconds after B sees the flash of A's gun he hears the sound of the bullet on the target. How far is each from the target?

6. A hears the sound of his bullet 5 seconds after he shoots and B hears the sound of his 6 seconds after he shoots. If A is between B and the target, how far apart are A and B?

7. B, who is 900 feet from the target, shoots over A's head, and A hears the report of the gun and the sound of the bulle at the same time. How far is A from the target?

8. In making a journey of 100 miles a man goes part of the way in an automobile at the rate of 16 miles an hour and has to walk the rest of the way. He completes the whole journey in 7 hours, and thinks that he walked at the rate of 4 miles an hour. How long must he have walked? How far must he have walked?

9. How far can a person who has 5 hours to spare ride in a carriage at the rate of 6 miles an hour so that he can return on foot at the rate of 4 miles an hour and arrive home in time?

10. An automobile is geared to run either 8 or 15 miles an hour. How long must the chauffeur use the 8-mile gearing in order to go 20 miles in just 2 hours?

11. A train passes a station at the rate of 10 miles an hour. Half an hour afterwards another train follows it at the rate of 12 miles an hour. How long will it take the latter to overtake the former?

12. A train passes a station at the rate of 10 miles an hour and c hours later another train follows it at the rate of 12 miles an hour. How long will it take the latter to overtake the former?

13. Solve problem 11 by making use of this result. Is there any advantage in solving a problem like this over solving one like problem 11?

14. A man in an automobile geared to 20 miles an hour starts to overtake another automobile that has 10 miles start and is geared to go 16 miles an hour. How long will it take the first machine to overtake the second one?

15. Two bodies move in the same direction with the velocity 12 and 18 feet a second respectively. The first has a start of a feet over the second. How long will it take the second one to overtake the first?

16. A bicycle policeman was riding along a street of a city at the rate of 5 miles an hour when an automobile overtook him. A minute later the automobile was a block ahead of him. There are 20 blocks to the mile at this place. Was the automobile exceeding the speed limit of 8 miles an hour?

17. At what time between 12 and 1 o'clock will the hands of a watch be opposite each other?

Hint. — If we take the minute space as the unit of distance and the minute as the unit of time, the velocity of the minute hand is 1 and that of the hour hand is $\frac{1}{12}$, since it passes over 5 minute spaces in 60 minutes.

18. At what time between six and seven will the minute hand be 15 minute spaces ahead of the hour hand?

19. At what time between 2 and 3 o'clock are the hands of a watch at right angles to each other?

20. The planet Venus makes the circuit of its orbit around the sun in 225 days and was in conjunction with the earth — that is, in the line between the earth and sun, — on Dec. 6, 1882. In how many days after this was the next conjunction? The earth makes a revolution around the sun in the same direction as Venus in 365 days.

21. The planet Jupiter goes around the sun in 4332 days. How long between conjunctions of Venus and Jupiter?

22. One boy can run around a circular track in 26 seconds, and another in 30 seconds. If they start at the same time and place and run in opposite directions, how soon will they meet?

23. A watch that loses 12 minutes in 24 hours is set at noon. What is the correct time when it is 5.30 P.M. by the watch?

24. A watch that gains 3 minutes every 24 hours is set at noon. What is the correct time when it is 9 P.M. by the watch?

25. A watch that gains 5 minutes every 24 hours is set at 9 Monday morning. What is the correct time Thursday morning when it is 7.20 by the watch?

26. A photograph was taken of a moving carriage, the time of exposure being $\frac{1}{8}$ of a second. The negative showed that during the exposure each spoke of one of the wheels had moved through an angle equal to a third of the angle between two consecutive spokes. What was the speed of the carriage, the wheel being 12 feet in circumference and having 12 spokes equal distances apart?

PROBLEMS WITH GEOMETRICAL APPLICATIONS

72. Formulæ from Geometry. — (*a*) If a is the area of a circle of radius r and c the circumference, then

$$a = \pi r^2,$$

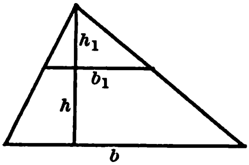
and

$$c = 2\pi r,$$

the approximate value of π (a Greek letter pronounced pī) being $3\frac{1}{7}$.

(*b*) If a triangle of base b and altitude h is cut by a line b_1 units long that is parallel to the base and h_1 units from the vertex, then

$$\frac{b}{h} = \frac{b_1}{h_1}.$$



(*c*) If V is the volume of a cylinder, B the area of its base, and h its height, then

$$V = Bh.$$

1. A certain triangle has a base 15 inches long and an altitude of 8 inches. How far above the base must a line parallel to the base be drawn in order that the part of it intercepted between the sides be 7 inches long?

2. A man wishes to determine roughly how far away a certain ship is. The ship is 100 feet long. He finds that by holding a piece of paper 6 inches long parallel to the side of the ship and 2 feet from his eye he can just cover the ship. How far away is the ship?

3. A man is standing in a room 6 feet back from a window which is 4 feet wide. On a river one half mile from the house a boat is moving at the rate of 5 miles an hour in a direction parallel to the side of the house. How long will the boat be in view of the man? How fast must the boat go in order to be in view 3 minutes?

Hint. — Let x = no. of minutes the boat is in view. Then in the preceding figure

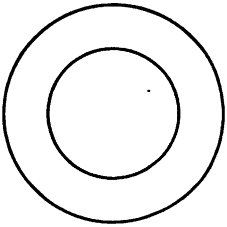
$$b = \frac{5 \cdot 5280}{60}x = 440\,x,\ b_1 = 4,\ h = 2646, \text{ and } h_1 = 6.$$

Caution. — In solving a problem always be sure that quantities of the same kind are expressed in terms of the same unit of measure. In the foregoing problem, for example, one distance is expressed in feet and another in miles. Express them both in feet. Also one time is expressed in hours and another in minutes. Express them both in minutes.

4. How far is the boat from the house if it is in view 2 minutes while going at the rate of 5 miles an hour?

5. A rectangular plot of ground three times as long as it is broad is to be laid out and surrounded by a walk 3 feet wide. What must be the length of the rectangle in order that the walk have an area of 993 square feet?

6. The ring between two circles with the same center is to be 1 inch wide and is to contain 12 square inches. What must be the radius of the inside circle?



7. In the triangle ABC, the angle B is twice as large as the angle A, and C is equal to the sum of A and B. How many degrees in each angle?

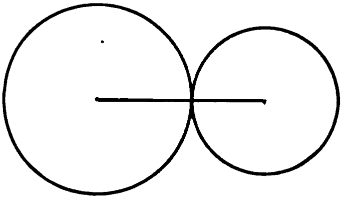
8. Two angles of a triangle are equal to each other and the third one is equal to two thirds the sum of the other two. What is the size of these two?

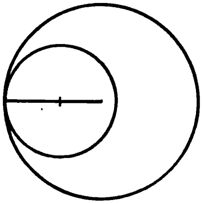
9. In the triangle ABC, the angle B is 10° larger than A, and C is 13° larger than B. How many degrees in each angle?

10. In the triangle ABC, A is 90° and B is four fifths of C. Find B and C.

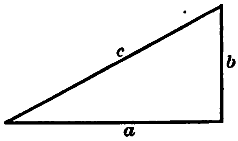
Facts from Geometry. — (*a*) Two circles are said to be tangent to each other if they have one and only one point in common.

(*b*) The line joining the centers of two such circles passes through their common point. The radius drawn to the point common to a circle and any of its tangents is perpendicular to the tangent.





(*c*) If a triangle has a right angle, the square of the side opposite this angle is equal to the sum of the squares of the other two sides. That is, $a^2 + b^2 = c^2$. The side opposite the right angle is called the **hypotenuse** of the triangle.



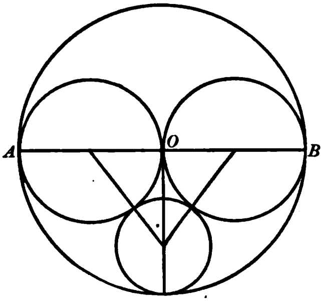
Historical Note. — The statement of this relation is called the *Pythagorean Theorem*, because Pythagoras, a Greek mathematician who lived in the sixth century B.C., was the first to prove it for all right-angled triangles. Many years before Pythagoras the Egyptians knew that if the legs of a right triangle were 3 and 4 respectively, the hypotenuse would be 5.

11. One side of a right-angled triangle is 2 inches long and the length of the hypotenuse is as much greater than 3 inches as the length of the third side is less than 5 inches. What is the length of the hypotenuse?

Hint. — Let $x =$ the length of the hypotenuse in inches. This exceeds 3 by $x - 3$ and therefore the length of the third side is $5 - (x - 3)$, or $8 - x$. Hence $x^2 = 4 + (8 - x)^2$.

12. One side of a right triangle is 3 inches and the hypotenuse is as much longer than 3 inches as the third side is shorter than 5 inches. What is the length of the hypotenuse?

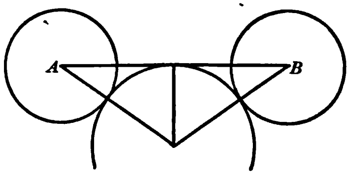
13. One side of a right triangle is 8 inches and the hypotenuse is 6 inches longer than the third side. What is the length of the third side?



14. On the diameter AB of the large circle two other circles are drawn with diameters AO and BO, O being the center of the large circle. The fourth circle is tangent to each of the other three circles, and the line connecting its center with O is perpendicular to AB. What is its radius, if $AB = 8$?

15. The distance AB between the centers of two equal circles of radius 5 inches is 20 inches. What is the radius of a third circle that is tangent to each of these circles and also to the line AB?

16. Suppose $AB = 2a$ and that the two given circles are of radius 6. Find the radius of the third circle.



17. Three points A, B, and C on a straight line have the abscissæ 3, 7, and 6 respectively. What must be the abscissa of a fourth point D in order that $\frac{DA}{CA} = 3\frac{DB}{CB}$?

18. What must be the abscissa of a fourth point P in order that $AC \cdot BP + BC \cdot AP = 0$?

19. What must be the abscissa of Q in order that $AQ \cdot BC - AB \cdot CQ = 0$?

20. The first of two cylindrical tanks contains 8 feet of water, and the second 10 feet. The bases are on the same level and have radii of 4 feet and 3 feet respectively. If the tanks should be connected, the water would assume the same depth in the two. What is this final depth?

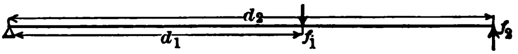
21. How large a ball can be sent by parcel post? (See Ex. 20, p. 45.)

APPLICATIONS OF THE LEVER

73. Law of the Lever. — The point of support of a lever is called the **fulcrum**. There are different kinds of levers, but in all of them two applied forces are in equilibrium when the product of the first force and the distance of its point of application from the fulcrum is equal to the product of the second force and the distance of its point of application from the fulcrum.

$$f_1d_1 = f_2d_2.$$

The two forces must be applied in the same direction when their points of application are on opposite sides of the fulcrum, as in a teeter board. The forces must be applied in the opposite direction when their points of application are on the same side of the fulcrum.



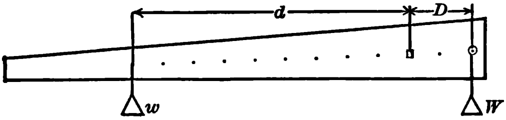
1. Two boys weighing 100 pounds and 80 pounds respectively are on the opposite ends of a teeter board and the board is balanced. How far is the second boy from the point of support if the first boy is 7 feet from it?

Hint. — Let x = distance in feet of the second boy from the point of support. Then $80x = 100 \cdot 7$.

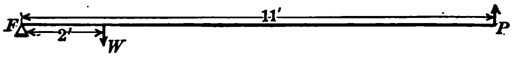
2. If the board in the preceding problem is 13 feet long, where should the point of support be?

3. If a boy weighing 90 pounds sits 6 feet from the point of support and is balanced by another boy 8 feet from the point of support, what is the weight of the second boy?

4. On a steelyard D is $1\frac{1}{2}$ inches and d is 8 inches. If w weighs half a pound, what is the weight of W when the steelyard is balanced?



5. How much force must be applied at P in order to raise a weight of 132 pounds at W?



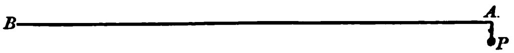
Hint. — First find what force at P will balance the weight. Then any greater force at P will raise the weight.

Further Facts about the Lever. — The effect of the weight of the lever is the same as if the entire weight were concentrated at the middle point.

When two forces f_1 and f_2 act in the same direction on a lever at the distances d_1 and d_2 respectively from the fulcrum and on the same side of it, the combined effect is $f_1d_1 + f_2d_2$.

6. What force must be applied at P to raise 116 pounds at W if we take into consideration the fact that the lever itself weighs 3 pounds to the foot?

7. AB is a uniform bar 4 feet long and weighs 8 ounces to the foot. Where must the fulcrum be put in order that the bar be balanced by P weighing 5 pounds?



8. Four weights of 1, 3, 4, and 6 pounds respectively are suspended from points on a straight lever 14 inches apart.

How far from the point of suspension of the first weight must the fulcrum be placed in order that the weights may be in equilibrium?

9. A and B carry a weight of 30 pounds suspended from a horizontal rod 8 feet long held between them. A is to lift 12 pounds and B 18 pounds. At what point of the rod should the weight be placed?

Hint. — In such a case as this the lifting force at one end multiplied by the distance of the weight from this end is equal to the lifting force at the other end multiplied by the distance of the weight from the end. The sum of the lifting forces is equal to the weight lifted. Hence if x = number of feet from A's end to the weight, $12x = 18(8 - x)$.

10. A and B are of the same height and sustain upon their shoulders a weight of 160 pounds placed on a pole 10 feet long. The weight is placed 6 feet from B. What is the weight sustained by each person?

11. The draft power of one horse is four fifths that of another. Where must the doubletree be divided in order that a load pulled by them be justly distributed?

12. Two men carry a load of 100 pounds on a pole between them. The pole is 10 feet long and the load is 7 feet from one end. How much does each man carry?

13. How much should a bar 10 feet long weigh per foot in order that it should balance about a point 2 feet from one end when a weight of 5 pounds is attached to this end?

14. A bar 10 feet long, weighing 2 pounds to the foot, has a weight of 6 pounds attached to one end. How far from one end must the point of support be in order that the bar may balance?

15. At what point must a bar 15 feet long and weighing 3 pounds to the foot be supported in order that it may balance when a weight of 8 pounds is attached to one end?

MISCELLANEOUS PROBLEMS

1. Find the cost per month of running the electric motor described in § 1.

2. An electric vacuum cleaner is used in a certain house that is lighted by electricity. In the month of September the cleaner was run 16 hours and the bill for electricity including lighting was $ 4.78. In the month of October, when the cleaner was run 20 hours and the lights were used 10 % more than in September, the bill was $ 5.33. What was the cost per hour of running the cleaner?

3. It was observed that two pipes feeding a cistern filled it in 15 minutes. At another time it was observed that one of the pipes alone filled it in 24 minutes. How long would it take the second pipe alone to fill it?

4. A tank containing 500 gallons is supplied by two pipes each of which can carry 10 gallons a minute. There is also an outlet pipe of unknown size. When all the pipes are open, the tank fills up in 30 minutes. How many gallons does the outlet pipe carry per minute?

5. Hiero, king of Syracuse, suspected that a supposedly gold crown which he had ordered contained some silver, and he called upon Archimedes to ascertain the composition of the crown. The latter found by experiment that a cubic inch of gold weighs 10.36 ounces, and that a cubic inch of silver weighs 5.85 ounces. He further found that the crown, which weighed 63 ounces, displaced 8.22 cubic inches of water. From these facts he determined the number of ounces of silver in the crown. How did he do it?

6. In the Fahrenheit thermometer, which is the one in common use in this country, the freezing point of water is marked 32° and the boiling point 212°. In the Centigrade thermometer, which is the one used in scientific work, these two points are marked 0° and 100° respectively. At what temperature will the two thermometers read the same?

7. A boy can mow the lawn in one and one half hours and his older brother can do it in an hour. The younger boy began to mow at ten o'clock and after he had been working awhile the other one took his place and finished the work at ten minutes after eleven. Which boy worked the longer? How much did each one mow?

8. A piece of platinum weighs twice as much as a piece of silver of the same size. A combination of the two metals containing three times as many cubic centimeters of silver as of platinum weighs fifteen ounces. How many cubic centimeters of platinum are there in the combination, if one cubic centimeter of silver weighs three eighths of an ounce?

Percentage. — From his study of percentage in arithmetic, the student has become familiar with the following facts:

$$i = prt, \tag{1}$$
$$a = p + i = p + prt, \tag{2}$$

where p = principal, or amount of money at interest, r = rate of interest, t = time, i = interest, a = amount, or principal plus the interest.

9. Solve equation (1) for p, r, and t in turn and interpret the results.

10. How long will it take $1000 at 5 % simple interest to amount to $2000?

11. What principal will amount to $650 in 5 years at 6 % simple interest?

12. A lawyer remitted to a client $475 after deducting his fee of 5 % for collecting a debt. How much did he collect?

13. An agent remitted to his principal $19,600 as the net proceeds from the sale of some property. His commission was 2 %. For how much did he sell the property?

14. What principal will amount to $1053 in 7 years at 5 % simple interest?

15. How long will it take $465 at 6 % simple interest to amount to $688.20?

CHAPTER VII

SYSTEMS OF LINEAR EQUATIONS. GRAPHS

74. Problems with Two Unknowns. — In problems like the following there are two unknown numbers to be determined.

Two weights on a lever balance when one is 12 inches and the other 16 inches from the fulcrum. But if the first weight is increased by 2 pounds, it is found that it must be placed 8 inches from the fulcrum to maintain the balance. What are the two weights?

Let $x =$ no. of pounds in the first weight,

and $y =$ no. of pounds in the second weight.

Then it follows from § 73 and the first statement of the problem that

$$12x = 16y. \tag{1}$$

Moreover $x + 2 =$ no. of pounds in the first weight after it has been increased by 2 pounds. Then

$$8(x + 2) = 16y. \tag{2}$$

If we equate the two values of $16y$ given in the equations (1) and (2), we get

$$12x = 8(x + 2).$$

Hence,

$$x = 4.$$

Substituting this value of x for x in equation (1), we get

$$16y = 48,$$

$$y = 3.$$

That is, the first weight is 4 pounds and the second one is 3 pounds.

It is to be observed that in this problem, instead of having a single description of one number, we have two descriptions, each stating a relation between two unknown numbers. These two descriptions are sufficient to identify the unknown numbers, since, translated into algebraic language, they are equivalent to equations (1) and (2) and there is only one pair of values of x and y for which these two equations are both true.

Neither of these two descriptions (or equations) alone would have been sufficient to identify the unknown numbers. Take, for example, the first one:

$$12x = 16y.$$

We could take for x any value whatever and for y three fourths of this value, and we should have a pair of numbers answering the description.

The student should verify that any pair of values of x and y in the following table answers the given description:

$x = 0,\ 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9,\ 10,\ 11,\ 12,\ 13,\ 14.$

$y = 0,\ \frac{3}{4},\ 1\frac{1}{2},\ 2\frac{1}{4},\ 3,\ 3\frac{3}{4},\ 4\frac{1}{2},\ 5\frac{1}{4},\ 6,\ 6\frac{3}{4},\ 7\frac{1}{2},\ —\ \ —\ \ —\ \ —$

Complete the table by filling in the missing values of y. Also assign some negative and fractional values to x and find the corresponding values of y.

This is a striking difference between descriptions of this kind and descriptions of single numbers such as were considered in the preceding chapter.

In the same way we can see that there is an unlimited number of pairs of values of x and y that answer description (2). But there is only one pair of values of x and y that answer both descriptions, and the problem is to find this one pair of values.

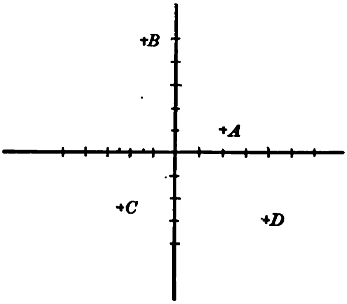
GRAPHICAL REPRESENTATION

75. A description of the relations between two numbers by means of an equation can also be given by means of a graphical representation.

The student is already familiar with the representation of numbers by points on a line.

A starting point, or origin, 0 is taken, and then every number is represented by the point whose distance from 0 is equal to the number. The representative point is to the right of 0 when the number is positive, and to the left when the number is negative.

In order now to represent a pair of numbers by means of a point we take two lines at right angles to each other and take their point of intersection as the origin on each line. If we call the numbers x and y, we lay off x on the horizontal line, in the way just recalled, and y in a similar way on the vertical line, counting distances above the origin as positive and those below as negative. Then through each of these points we draw a line perpendicular to the line on which the point lies. The point of intersection of these two perpendiculars is the point representing the pairs of values of x and y that were used. In the figure given above the points A, B, C, and D represent the pairs of values $2, 1$; $-\frac{3}{2}, 5$; $-\frac{5}{2}, -\frac{5}{2}$; and $4, -3$ respectively.



76. Definitions.—If the numbers x and y determine the point A in the way just described, x is called the **abscissa** of the point and y its **ordinate**. The two together are called the **coördinates** of the point.

The abscissa of a point is always given first. Thus the point (2, 3) is the point whose abscissa is 2 and ordinate 3.

The two perpendicular lines are called the **axes of coördinates**, or **coördinate axes**.

The axis on which the abscissæ are laid off is called the **x-axis**, and the one on which the ordinates are laid off is called the **y-axis**. The x-axis is usually horizontal.

Every pair of values of x and y is represented by a point, and every point represents a pair of values of x and y. That is, every point has two coördinates. These cöordinates can be found by drawing perpendiculars from the point to the axes and measuring the distances from the origin to the feet of these perpendiculars.

EXERCISES

For all graphical work in this book cross-ruled paper should be used whenever it is available. Its use is not essential, however, if the student will exercise proper care in measuring his distances accurately. He should be on his guard against falling into the habit of slovenly and inaccurate work.

Draw two axes and locate the following points, using $\frac{1}{2}$ inch or 1 centimeter as the unit distance.

1. $(2, 3)$, $(5, -2)$, $(0, 1)$.
2. $(-2, -2)$, $(-3, -3)$, $(-\frac{5}{2}, -\frac{5}{2})$, $(1, 1)$, $(4, 4)$.
3. $(-\frac{3}{2}, \frac{3}{2})$, $(0, 0)$, $(3, -3)$, $(\frac{5}{2}, -\frac{5}{2})$.
4. $(7, 0)$, $(5, 0)$, $(4, 0)$, $(1, 0)$, $(-3, 0)$, $(-8, 0)$.
5. $(0, -10)$, $(0, -8)$, $(0, -7)$, $(0, -4)$, $(0, 1)$, $(0, 2)$, $(0, 3)$.
6. $(0, 5)$, $(3, 4)$, $(4, 3)$, $(5, 0)$, $(-4, 3)$, $(3, -4)$.
7. $(1, -2)$, $(3, -6)$, $(-1, 2)$, $(4, -8)$, $(-\frac{1}{2}, 1)$.
8. $(0, 2)$, $(-1, 1)$, $(1, 3)$, $(4, 6)$, $(-3, -1)$.
9. $(-1, 3)$, $(2, 0)$, $(1, 1)$, $(3, -1)$, $(-2, 4)$.
10. $(-1, -2)$, $(3, 6)$, $(1, 2)$, $(-4, -8)$; $(5, -10)$.

11. See if the points of each of the Exercises 2–5 lie on a straight line.

12. See if the points of each of the Exercises 7–10 lie on a straight line.

13. Do the points of Ex. 6 lie on a straight line?

Translate each of the following statements into an equation, using the letter x to represent the abscissa, and the letter y the ordinate, of a point.

14. The abscissa of a point equals its ordinate.

15. The abscissa of a point equals twice its ordinate.

16. Three times the abscissa of a point equals four times its ordinate.

17. The ordinate of a point is 2 greater than its abscissa.

18. The ordinate of a point is 5 less than its abscissa.

19. The ordinate of a point is 3 greater than twice its abscissa.

20. The sum of the abscissa and the ordinate of a point is 0.

21. The abscissa of a point is 2 less than four times its ordinate.

22. The sum of one half the abscissa of a point and one third its ordinate is 1.

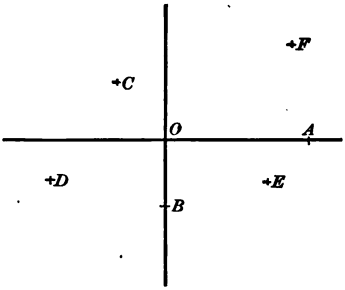
23. Draw a line parallel to the x-axis and 3 units above it, and measure the coördinates of 5 points on it.

24. Draw a line parallel to the x-axis and 3 units below it, and measure the coördinates of 5 points on it.

25. Draw a line parallel to the y-axis and 2 units to the left of it, and measure the coördinates of 5 points on it.

26. Draw a line parallel to the y-axis and 2 units to the right of it, and measure the coördinates of 5 points on it.

27. Draw a line through the origin making an angle of 45° with the x-axis and measure the coördinates of 5 points on it.



28. Draw a line through the origin making an angle of 135° with the x-axis and measure the coördinates of 5 points on it.

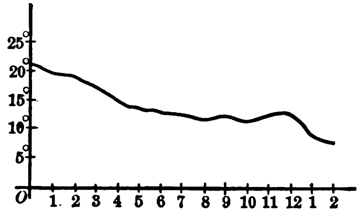
29. Determine by accurate measurements the abscissæ of the points A, B, C, D, E, and F in the accompanying figure, using $\frac{1}{2}$ inch as the unit.

30. The records of the U. S. weather bureau at New York City show the following temperatures for January 12, 1912:

	Degrees
12 Midnight	22
1 A.M.	20
2 A.M.	19
3 A.M.	17
4 A.M.	15
5 A.M.	14
6 A.M.	13
7 A.M.	13
8 A.M.	12
9 A.M.	13
10 A.M.	12
11 A.M.	13
12 M.	13
1 P.M.	9
2 P.M.	7

We can represent graphically the temperature at any of these hours by plotting the point whose abscissa is the time, measured in hours from midnight, and whose ordinate is the temperature, measured in degrees. Thus the point(5, 14) represents the temperature at 5 A.M., as given in the table.

If we represent all these temperatures on the same figure and connect the points by straight lines, or by a smooth curve, we shall have before us a graphical representation of the variations of the thermometer in the period from midnight to 2 P.M. This representation seems to give more information than is contained in the table. For example, the ordinate of the point on the curve, or broken line, whose abscissa is 4½ apparently gives the temperature at half-past four. But as a matter of fact it merely represents a guess on our part as to what this temperature probably was. And this guess was based on the supposition that the change of temperature from one hour to the next hour was a gradual one. But we had no definite information as to this.



The drawing then is accurate at certain points, and is probably approximately correct at the intermediate points. It presents to the eye at once information that could be obtained from the table only by a more careful study.

This method of graphical representation is useful in the presentation of statistical data and is much used for this purpose.

It is not necessary that the unit of length on the axis of ordinates be the same as the unit of length on the axis of abscissæ. These lengths should be determined by considerations of convenience.

Exhibit graphically the information contained in the following tables:

31. Temperatures for January 13, 1912.

	Degrees		Degrees
12 Midnight	5	8 A.M.	−1
1 A.M.	4	9 A.M.	−1
2 A.M.	3	10 A.M.	2
3 A.M.	2	11 A.M.	1
4 A.M.	1	Noon	5
5 A.M.	0	1 P.M.	6
6 A.M.	−2	2 P.M.	9
6.30 A.M.	−3	3 P.M.	10
7 A.M.	−2		

32. Monthly exports of copper in thousand tons for 1909, 1910, and 1911.

	1909	1910	1911
January	19	27	29
February	14	25	19
March	21	20	24
April	28	13	27
May	31	21	28
June	34	23	30
July	35	23	37
August	23	28	28
September	20	32	26
October	24	28	22
November	24	30	26
December	28	32	37

Let the unit of abscissæ represent one month and the unit of ordinates represent 5000 tons. Plot a separate curve for each of the years, referring all three to the same set of axes, so as to facilitate a comparison of business conditions for the three years.

33. A week's record of the temperatures of a hospital patient, the temperatures having been taken every 12 hours.

	Morning	Evening
Monday	101.5	102
Tuesday	101.5	101.2
Wednesday	98.1	103.5
Thursday	97.2	103.6
Friday	101	103.2
Saturday	102.6	101.4
Sunday	98.8	101.5

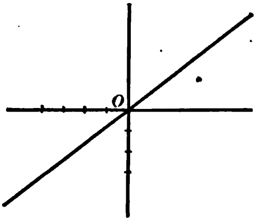
34. World's gold production in millions.

1892	147	1902	297
1893	157	1903	328
1894	181	1904	347
1895	199	1905	380
1896	202	1906	403
1897	236	1907	413
1898	287	1908	443
1899	287	1909	454
1900	255	1910	452
1901	261	1911	466

Let the unit of ordinates represent 20 millions.

35. Lay off the points whose coördinates are the pairs of values of x and y given in the table of § 74. Can you draw a straight line through all these points?

77. As a matter of fact there is a straight line such that the ordinate of every point on it is three fourths of the abscissa.



The line in the accompanying figure is this line.

If we call the line l and take any point above l, its ordinate is more than three fourths of its abscissa; and if we take any point below l, its ordinate is less than three fourths of its abscissa. In other words, the ordinate of any point on l is three fourths of its abscissa, and the ordinate of any point not on l is different from three fourths of its abscissa.

We express these facts by saying that the line l is the *locus* of the equation $12x = 16y$, or that $12x = 16y$ is the *equation* of the line.

Since an unlimited number of values of the unknown numbers answer a description of the kind we have been considering, these numbers are called **variables.**

Definition. — An equation in the variables x and y is said to be of the **first degree** if each member is either a constant or a polynomial of the first degree in the variables.

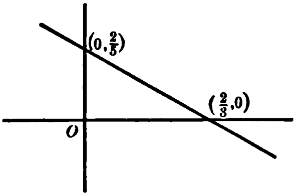
Examples. $3x - 5y + 2 = 0$, $4x + y = 7$, $8x = 5$, $3y + 8 = 0$, and $6x + 5y + 1 = 3x - 4y + 2$ are equations of the first degree.

Historical Note. — It was stated on p. 61 that most of the subject matter of this book has been known for thirteen centuries. The method of graphical representation of the relation between two variables that has just been described is an important exception to this. The method was invented in the seventeenth century by René Descartes, a French philosopher and mathematician.

It was by means of this graphical representation that European mathematicians obtained a clear notion of the nature of negative numbers.

78. Linear Equations. — It is proved in Analytic Geometry that the locus of an equation of the first degree in one or two variables is a straight line. It is because of this fact that equations of the first degree are called **linear** equations.

Since a straight line can be drawn when two of its points are known, it is easy to draw the locus of an equation of the first degree. Suppose, for example, we wish to draw the locus of the equation $3x + 5y - 2 = 0$. In this equation, if $x = 0$, we must have $y = \frac{2}{5}$; and if $y = 0$, we must have $x = \frac{2}{3}$. Hence the locus passes through the points $(0, \frac{2}{5})$ and $(\frac{2}{3}, 0)$. In general, in order to draw the straight line given by an equation, it is best to find where the line cuts the axes and then draw the straight line through these points. Since the ordinate of every point on the x-axis is 0, we can find the abscissa of the point where the line cuts the x-axis by putting y equal to 0 in the equation and solving the equation for x. If the line passes through the origin, we cannot use this method for drawing it.

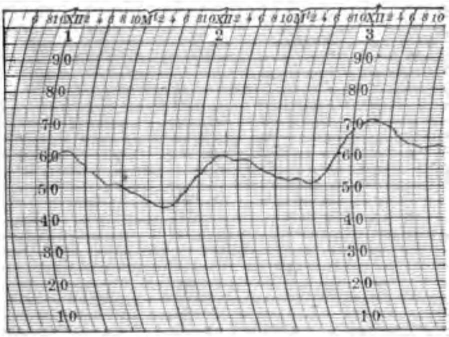


Why? How can we draw such a line when its equation is given?

The coördinates x and y of any point P on a locus answer the description given by the equation, and the coördinates of any point not on the locus do not answer this description. We can therefore look upon the locus as a pictorial, or graphical, description of the same relation between x and y as that given by the equation.

We shall see later that there are many equations whose loci are not straight lines but are circles and other kinds of curves.

79. When we have a description of a relation between two variables given by means of an equation, we can get the description in graphical form by constructing the locus of the equation. But we frequently get descriptions in graphical form that are not expressed by means of equations. The accompanying figure is such a description. It describes the relation between the time and the temperature and is obtained by the use of an apparatus which moves a sheet of paper at a uniform rate under a pen which is attached to one end of a rod. As the temperature changes, the rod moves on a pivot at its other end. This causes the pen to describe the arc of a circle, and it is for this reason that the ordinates in this figure are arcs of circles instead of straight lines.



We shall confine our attention to descriptions that can be expressed by means of equations.

80. Graphical Solution of a System of Equations. — We can utilize the graphical representation of equations for the purpose of solving a system of two equations of the first degree in x and y. Suppose, for example, that we wish to solve the equations

$$3x + 2y = 12, \qquad y - x = 1.$$

We draw the locus of each equation and measure the coördinates of the point of intersection of these loci. The abscissa of this point is the number represented by x, and the ordinate is the number represented by y, in the descriptions given by the equations.

EXERCISES

Solve each of the following systems of equations graphically:

1. $2x - 3y + 4 = 0$,
$x + y = 3$.

2. $5x + 4y = 2$,
$2x + y + 1 = 0$.

3. $4x + y = 2$,
$6x - 3y = 3$.

4. $x + y = 0$,
$x - y = 0$.

5. $3x = 4y$,
$6y - 3 = 0$.

6. $x + y = 1$,
$x - y = 1$.

7. $x - 5 = 0$,
$y + 2 = 0$.

8. $2x + 3y = 1$,
$4x + 6y = 5$.

9. $x + y = 5$,
$y - x = 1$.

10. $3x - 2y - 1 = 0$,
$5x + 6y = 11$.

11. $7x = 5y$,
$2x + 3y = 0$.

12. $x - y + 2 = 0$,
$4y + 5 = -9x$.

13. $6x + 3 = 2y$,
$y - 2x - 2 = 0$.

14. $3x + 6 = 0$,
$8y - x = 0$.

15. $5y - 3x = 1$,
$7y + 4x = 26$.

16. $y = 3x + 5$,
$x + y = 5$.

17. $x + 4y - 3 = 0$,
$y - x = 5$.

18. $2x + 4y = 7$,
$3x + y = 8$.

19. $y + x + 4 = 0$,
$9y - 2x + 14 = 0$.

20. $\frac{1}{5}x + 2y = 1$,
$3x + 5y = 15$.

21. $\frac{1}{3}y - \frac{1}{4}x = 1$,
$\frac{2}{3}y + \frac{3}{4}x + 3 = 0$.

22. $2y = 3x$,
$y - x - 1 = 0$.

23. $2y - 10 = 0$,
$x + 3 = 4$.

24. $x - 2y = 11$,
$4x + 3y = 0$.

25. $5x + y = -9,$
$x + y + 1 = 0.$

26. $x + y = 6,$
$x - y = 0.$

27. $5x - y = 5,$
$3x + 2y = 16.$

28. $y - x = -3,$
$3x + 4y = 23.$

29. $5x - 2y + 13 = 0,$
$x + 1 = 0.$

30. $x + y + 9 = 0,$
$2x + y + 12 = 0.$

31. $y - x + 3 = 0,$
$2x - y = 0.$

32. $5x - 2y = 4,$
$3x + 3y + \frac{3}{4} = 0.$

33. $\frac{3}{2}x + \frac{5}{4}y = 1\frac{1}{4},$
$3x + 2y = 5.$

34. $x + y = 8,$
$3x - 2y = 4.$

35. $5x - 3y - 19 = 0,$
$y + x + 1 = 0.$

36. $3x = -1 - y,$
$2y + 3x + 8 = 0.$

81. Solution by Substitution. — There are two other methods in general use for solving a system of two equations of the first degree in two unknowns. Both are analytic in their nature rather than graphical. The first may be described as the **Method of Substitution.**

The application of this method is illustrated in the solution of the following equations:

$$2x + 5y = 3, \qquad (1)$$

$$3x + 2y = 4. \qquad (2)$$

Solving equation (2) for x, we get

$$x = \tfrac{1}{3}(4 - 2y). \qquad (3)$$

Using this value for x in (1), we get

$$\tfrac{2}{3}(4 - 2y) + 5y = 3, \qquad (4)$$

or

$$\tfrac{1}{3}(8 - 4y) + 5y = 3. \qquad (5)$$

Clearing of fractions,

$$8 - 4y + 15y = 9, \qquad (6)$$

$$11y = 1, \qquad (7)$$

$$y = \tfrac{1}{11}. \qquad (8)$$

Hence,

$$x = \tfrac{1}{3}(4 - 2y) = \frac{4 - \frac{2}{11}}{3} = \frac{42}{33} = \frac{14}{11}. \qquad (9)$$

Check. — Substitute these values in (1) and (2).

$$\tfrac{28}{11} + \tfrac{5}{11} = 3,$$
$$\tfrac{42}{11} + \tfrac{2}{11} = 4.$$

Since these are true statements, it follows that equations (1) and (2) are both satisfied when we put $\frac{14}{11}$ for x and $\frac{1}{11}$ for y. Our work is therefore correct. If one or both of these statements had been untrue, we could have inferred that there was a mistake somewhere in our work.

Rule. — *Solve either equation for one of the unknowns in terms of the other one.*

In place of this unknown in the other equation substitute the value of it thus obtained, and solve the resulting equation for the other unknown.

Substitute the resulting value of the second unknown in any one of the preceding equations that contains both unknowns and solve the resulting equation for the first unknown. In doing this select the simplest equation you can.

Check your result.

82. Solution by Addition or Subtraction. — The second method referred to in § 81 is that of **Addition or Subtraction**. The application of it is illustrated in the solution of the following equations:

$$7x - 4y = 2, \tag{1}$$
$$2x + 3y + 4 = 0. \tag{2}$$

Multiply each member of (1) by 3 and each member of (2) by 4.

$$21x - 12y = 6, \tag{3}$$
$$8x + 12y = -16. \tag{4}$$

The sum of the two left members of (3) and (4) must equal the sum of the two right members. Hence

$$29x = -10, \tag{5}$$
$$x = -\tfrac{10}{29}. \tag{6}$$

Substitute this value of x for x in (2) and solve the resulting equation for y.

$$-\tfrac{20}{29} + 3y + 4 = 0, \tag{7}$$
$$-20 + 87y + 116 = 0, \tag{8}$$
$$87y = -96, \tag{9}$$
$$y = -\tfrac{96}{87} = -\tfrac{32}{29}. \tag{10}$$

Check. — Substitute the values of x and y given by (6) and (10) in (1) and (2).

$$-\tfrac{70}{29}+\tfrac{128}{29}=2,$$
$$-\tfrac{20}{29}-\tfrac{96}{29}+4=0.$$

These are true statements, and our work is therefore correct.

From the original equations, each of which contains both x and y, we obtained (5) which contains only x. In such cases we say that we have *eliminated* y. We could have solved the equations by first eliminating x. If in a given problem it is easier to eliminate one of the unknowns rather than the other, the student should follow the easier course.

Rule. — *Decide which unknown can be eliminated the more easily.*

Multiply each member of the first equation by a number and each member of the second equation by another number, such that the coefficients of this unknown in the two resulting equations have the same absolute value.

Form a new equation, each of whose members is the sum or the difference of the corresponding members of the last equations according as these coefficients have unlike or like signs.

The resulting equation will contain only one unknown. Solve for this unknown.

Substitute the resulting value of this unknown in either of the preceding equations that contains both unknowns and solve the resulting equation for the second unknown.

Check your result.

It is sometimes more convenient to find the value of the second unknown by going back to the original equations and eliminating the first unknown.

83. Dependent and Inconsistent Equations. — Sometimes the second equation contains no information concerning the relation between the numbers that is not contained in the first equation. Consider, for example, the equations

$$2x+5y-1=0,$$

and

$$4x+10y=2.$$

Every pair of values of x and y that satisfy the first of these equations also satisfy the second. The student should test this statement by finding a number of pairs of values of x and y that satisfy the first equation and then showing that these values also satisfy the second equation.

Equations related in this way are said to be **dependent.** If each contains information not contained in the other, the equations are said to be **independent.**

Two dependent equations have the same locus.

Sometimes there are no pairs of values of x and y that satisfy both equations.

If, for example, we used the method of Addition or Subtraction to solve the equations

$$2x + 3y = 2$$

and

$$4x + 6y = 1,$$

we should get

$$0 = -3.$$

Now this impossible relation was obtained on the supposition that there were values of x and y that satisfy both equations. We must conclude therefore that there are no such values.

Equations related in this way are said to be **inconsistent.**

The loci of inconsistent linear equations in two variables are parallel lines.

EXERCISES

Solve the systems of equations in the odd Exercises by the Method of Substitution, and those in the even Exercises by the Method of Addition or Subtraction. Check your results.

Also solve each system graphically and compare your results with those obtained analytically. Graphical solutions cannot be expected to be exact.

1. $2x + 4y = 16,$
$x - y + 1 = 0.$

2. $4x + 5y = 8,$
$3x + 2y = 6.$

3. $5x - 2y + 3 = 0,$
$x - y = 0.$

4. $2x - 3y - 1 = 0,$
$4x - 3y = 2.$

5. $y - 2 = 3(x - 1),$
$y + 1 = 1 + x.$

6. $4y + 2 = 5(x + 3),$
$y - 4 = 2(x - 3).$

7. $y - 1 = -4(x + 2),$
$y - 1 = 2(x + 2).$

8. $\frac{1}{4}x - \frac{1}{5}y = 1,$
$\frac{1}{2}x + \frac{1}{3}y = 2.$

9. $y + 3 = x - 1,$
$y - 2 = x + 3.$

10. $\frac{2}{3}(x - 1) + \frac{3}{2}(y + 1) = 1,$
$x + 2y + 1 = 0.$

11. $\frac{1}{3}x - \frac{1}{2}y = 1,$
$2x = 4 + 3y.$

12. $y = x + 1,$
$3x + 2y = 7.$

13. $5x - 2y = 14,$
$x + 4y = 16.$

14. $2y - 5x = 5,$
$y - 2x = 2.6.$

15. $7x + 2y = 1,$
$5 + y = 2x.$

16. $7x + 13y = 25,$
$4x - 11y = -41.$

17. $9x - 1.1y = 0,$
$5x + 1.6y = 0.$

18. $3x - 2y = 15,$
$-\frac{5x}{3} + 8y = -8\frac{1}{3}.$

19. $x - y = 3,$
$.11x - 10y = 30.$

20. $\frac{2}{3}x + \frac{1}{2}y = \frac{77}{36},$
$\frac{1}{4}x + \frac{1}{3}y = \frac{7}{6}.$

21. $2x + .3y = 5,$
$.4x + 6y = 6.94.$

22. $13x + 20y = 124,$
$-8x + 13y = -51.$

23. $y = 5x + 2,$
$y - 1 = 3(x - 2).$

24. $.3y + .2x + 1 = 0,$
$3x + .4y - 5.5 = 0.$

25. $4x + 6 = 0,$
$3y - 9 = 0.$

26. $.5x + .2y = 1.7,$
$.35x + .4y = 16.4.$

27. $x = y + 5,$
$x = 5 - y.$

28. $y + 2 = \frac{1}{2}(x - 3),$
$y + 2 = -\frac{2}{3}(x - 3).$

29. $x + y = 5\frac{1}{2},$
$2x + 3y = \frac{23}{2}.$

30. $2x + 5y = 3,$
$4x + 10y = 6.$

31. $4x - y + 3 = 0,$
$5x + 2y = 12\frac{1}{2}.$

32. $y + \frac{1}{3} = \frac{1}{3}(x + 1),$
$y - 4 = -2(x + \frac{1}{4}).$

84. In some of the following problems, one of the relations between the unknowns is so simple that the second unknown can be expressed immediately in terms of the first one. When this is done, the problem leads to but one equation from which to determine the first unknown. Some of the problems in the preceding chapter are of this kind.

PROBLEMS

1. Two weights on a lever balance when one is 10 inches and the other 8 inches from the fulcrum. If the first weight increased by 3 pounds is placed 9 inches from the fulcrum, the balance is maintained. Find the two weights.

2. Two weights balance when one is 12 inches and the other 15 inches from the fulcrum. The balance is maintained if the first weight is moved 2 inches further from the fulcrum and half a pound is added to the second. Find the weights.

3. Two weights of 30 and 40 pounds respectively balance when resting on a lever at unknown distances from the fulcrum. If 10 pounds are added to the first weight, the other one must be moved 3 feet further from the fulcrum in order to maintain the balance. What was the original distance from the fulcrum to each of the weights?

4. Two weights of w_1 and w_2 pounds respectively balance when resting on a lever at unknown distances from the fulcrum. If 3 pounds is added to the first weight, the other one must be moved b feet further from the fulcrum in order to maintain the balance. What was the original distance from the fulcrum to each of the weights?

5. Two boys A and B wish to determine their respective weights by means of a teeter board and a weight of 10 pounds. They find that they balance when A is 11 feet and B is 9 feet from the fulcrum. If A places the weight on the board with him, it is necessary for B to move 1 foot further from the fulcrum in order to maintain the balance. What are their weights?

6. Two boys A and B wish to determine their respective weights by means of a teeter board and a weight of c pounds. They find that they balance when A is a feet, and B is b feet, from the fulcrum. If A places the weight on the board with him it is necessary for B to move f feet further from the fulcrum in order to maintain the balance. What are their weights?

7. Where must the fulcrum be placed in order that two weights of 8 and 15 pounds respectively balance when placed at opposite ends of a lever 12 feet long?

8. Where must the fulcrum be placed in order that two weights of 10 and 14 pounds respectively balance when placed at opposite ends of a lever a feet long?

9. A man has $8000 to invest in two enterprises. The first yields $5\frac{1}{2}\%$ and the second 5%. How much must he put in each enterprise in order that his annual income from the two may be $425?

10. A part of a dollars is invested at 4% and the remainder at 5%. How much must be in each investment in order that the total income may be b dollars?

The Decimal System of Notation. — In our decimal system of notation for numbers the importance of a digit depends upon its relative position. Thus in 47 the 4 stands for 4×10 because it stands in the tens' place, whereas in 54 the 4 stands for 4×1 because it stands in the units' place. The integer of two digits whose tens' digit is x and whose units' distance is y is accordingly $10x + y$, and not xy as might be supposed.

11. Write the integer whose units' digit is x and whose tens' digit is y.

12. Write the integer whose units' digit is y and whose tens' digit is twice as great.

13. The sum of the digits of a certain integer between 10 and 100 is 7, and when the digits are reversed, the number is decreased by 27. What is the number?

14. The difference of the digits of a certain integer between 10 and 100 is 2, and the sum of the digits is 9 less than the number. What is the number?

15. A man rows 12 miles downstream in 2 hours and returns in 3 hours. How fast does he row in still water and what is the rate of the current?

Hint. — His rate downstream is equal to his rate in still water plus the rate of the current, and his rate upstream is equal to the difference of these rates.

16. A crew can row 7 miles an hour downstream and 2 miles an hour upstream. What is the speed of the current?

Hint. — In this problem there are two unknown numbers, although only one of them is asked for. In such a case proceed as if both of the unknowns were asked for, but try to eliminate the one not desired.

17. A man rows a miles downstream in 2 hours and returns in 3 hours. How fast does he row in still water and what is the rate of the current?

Are your results in Problems 15 and 17 consistent?

18. If a crew can row a miles an hour downstream and b miles upstream, what is the rate of the current?

Are your results in Problems 16 and 18 consistent?

19. A passenger train running 50 feet a second passes a freight train on a parallel track in 30 seconds. The freight is running in the same direction as the passenger and at the rate of 30 feet a second. A short time afterwards the passenger comes to a stop and the freight overtakes it. It is 8 seconds from the time the front of the freight engine is even with the rear of the passenger train until the two engines are abreast. How long is each train?

20. A passenger train running p feet a second passes a freight train on a parallel track in m seconds. The freight is running at the rate of f feet a second in the same direction as the passenger. A short time afterwards the passenger comes to a stop and the freight overtakes it. It is n seconds from the time the front of the freight train is even with the rear of the passenger train until the two engines are abreast. How long is each train?

21. A passenger train running 54 feet a second passes a freight train on a parallel track in 28 seconds. The freight is running in the same direction as the passenger and at the rate of 32 feet a second. A short time afterwards the passenger slows down to 25 feet a second and the freight overtakes it. It is 30 seconds from the time the front of the freight engine is even with the rear of the passenger train until the two engines are abreast. How long is each train?

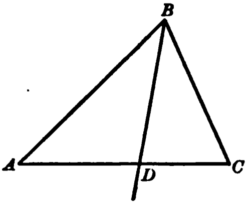
22. The lengths of two trains are 200 feet and 180 feet respectively. When running on parallel tracks at certain rates in opposite directions they passed each other in $4\frac{2}{9}$ seconds. At another time when running at the same respective rates in the same direction the second one passed the first one in 38 seconds. Find the rates at which they ran.

23. The lengths of two trains are a and b feet respectively. When they run on parallel tracks at certain rates of speed in opposite directions they pass each other in 6 seconds. When they run at the same respective rates in the same direction they pass in 12 seconds. Find the rates at which they run.

24. Three bicyclists A, B, and C are riding on a half-mile track. A and B are going in opposite directions at the same rate, while C is going in the same direction as B but at a different rate of speed. A and C pass each other every 40 seconds, while B and C pass each other every 2 minutes. Find their rates.

Property of a Triangle. — If a line divides an angle into two equal angles it is said to *bisect* the angle. For example, the line BD bisects the angle ABC.

It is shown in geometry that if the line BD bisects the angle ABC of the triangle, then $\frac{AD}{AB}=\frac{DC}{BC}$, or $AD \cdot BC = AB \cdot DC$.



25. If $AB=12$, $BC=8$, $CA=9$, what are the lengths of AD and DC?

26. Find AD and DC in the case $AB=5$, $BC=4$, $CA=b$.

27. Find AD and DC in the case $AB=m$, $BC=m$, $CA=n$.

28. Represent graphically the relation between the readings of a Fahrenheit and of a Centigrade thermometer. See Problem 6, p. 100.

29. By means of the graph of Problem 28 determine the Centigrade readings for the temperatures given in Exercise 30, § 76.

30. What is the Centigrade reading when the temperature is zero Fahrenheit?

31. In the Réaumur thermometer the zero is at the same point as in the Centigrade and the temperature at which water boils is marked 80°. Represent graphically the relation between the readings of a Fahrenheit and of a Réaumur thermometer.

32. By means of the graph of Problem 31, determine the Réaumur readings for the temperatures given in Exercise 30, § 76.

33. Determine graphically the Réaumur reading corresponding to a Centigrade reading of 15°.

34. Solve Problem 14, § 71, graphically.

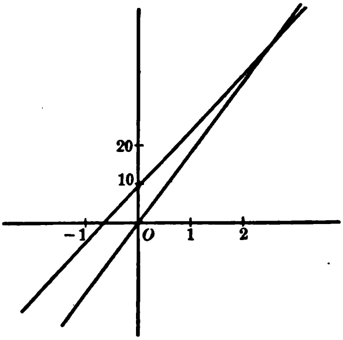
Suggestion. — Represent the time in hours on the axis of abscissæ and the distance in miles on the axis of ordinates. Then the line that represents graphically the distance traveled by the first mentioned automobile in a given time is the locus of the equation

$$d = 20\,t.$$

The corresponding line for the other machine is the locus of the equation

$$d = 16\,t + 10,$$

since it travels at the rate of 16 miles per hour and had gone 10 miles before the machine first mentioned started. The abscissa of the point of intersection of these two loci is the time (in hours) it takes the faster machine to overtake the other one.



In drawing this figure we took the unit of length on the axis of ordinates shorter than that on the axis of abscissæ.

35. Solve graphically Problem 11, § 71.

36. Determine graphically how long it will take an automobile going 25 miles an hour to overtake another one that had a start of 15 miles and is going at the rate of 16 miles an hour.

37. Determine graphically how long it will take a man walking $4\frac{1}{2}$ miles an hour to overtake another man who had an hour's start and is walking at the rate of 4 miles an hour.

85. Systems of Three Linear Equations in Three Unknowns. — Two descriptions of the relations connecting three unknown numbers are not sufficient to identify these numbers. For example, there are many sets of values of x, y, and z that answer the two following descriptions:

$$3\,x + y + z = 4, \tag{1}$$

$$x - 2\,y - 3\,z = 1. \tag{2}$$

Find five sets of values of x, y, and z that answer these descriptions.

But there is only one set of values of the three unknowns that answer three descriptions of the relations connecting them. Consider, for example, the description

$$x + y + 2z = 5 \tag{3}$$

in connection with (1) and (2).

If we eliminate y between (1) and (3), we get

$$2x - z = -1. \tag{4}$$

And if we eliminate y between (2) and (3), we get

$$3x + z = 11. \tag{5}$$

From (4) and (5) we find that

$$x = 2, z = 5.$$

Finally, from (3) we get $y = -7$.

If we substitute these values of x, y, and z for the respective unknowns in equations (1), (2), and (3), we get

$$6 - 7 + 5 = 4,$$
$$2 + 14 - 15 = 1,$$
$$2 - 7 + 10 = 5.$$

Since these are true equations, it follows that equations (1), (2), and (3) are satisfied when we put 2 for x, -7 for y, and 5 for z; and these are therefore numbers that answer the original descriptions.

It was not necessary to proceed in exactly this way. We might have chosen a different unknown for the first elimination and accordingly have combined the original equations differently. But the same variable must always be eliminated in the first two eliminations.

From what has been said the student should write out in detail a rule for the solution of a set of three equations of the first degree in three unknowns.

86. Dependent and Inconsistent Equations in Three Unknowns. — Sometimes the three equations are such that one of them gives no more information concerning the relations connecting the unknowns than is given by the other two equations. The equations

$$2x + 3y + 4z = 5, \tag{1}$$
$$-x + 2y - 3z = 2, \tag{2}$$
$$x + 5y + z = 7, \tag{3}$$

form such a set, inasmuch as the members of (3) are the sums of the corresponding members of (1) and (2), and any set of values of x, y, and z satisfying (1) and (2) must also satisfy (3).

The student should give to z in (1) and (2) any convenient value and solve the resulting equations for x and y. He will then find that these values of x, y, and z satisfy (3).

It is possible for the equations to be such that there are no values of the unknowns satisfying all of them. The equations

$$\begin{aligned} 2x + 3y + 4z &= 5, \\ -x + 2y - 3z &= 2, \\ x + 5y + z &= 1, \end{aligned}$$

are of this kind, inasmuch as the left member of the third equation is the sum of the left members of the other two equations, while the right member of the third is not the sum of the right members of the other two. (See § 83.)

The student should see what will happen if he applies to such a set of equations the rule he has just formulated for the solution of a set (or system) of three equations of the first degree in three unknowns.

87. In some of the following exercises the student is asked to solve systems of four equations in four unknowns. He should derive a rule of procedure in such cases by analogy from the rule for the solution of systems of three equations in three unknowns.

EXERCISES

Solve the following systems of equations and check your results:

1. $x + 2y + z = 0,$
$x - 2y - 8 = 0,$
$x + y + z = 3.$

2. $x + y + z - 1 = 0,$
$x - y - z = 1,$
$2x + 3y = 8.$

3. $x + y + 2z = 0,$
$2x - y - 2z - 1 = 0,$
$3x + 6z - 5y + 1 = 0.$

4. $2x - 3y + z + 1 = 0,$
$5x + z - 1 = 0,$
$23x + 3y + 4z = 2.$

5. $2l - 3n + m = 6,$
$l + m + n - 1 = 0,$
$l - 4m - 4n = 1.$

6. $x + y = 5,$
$y + z = 8,$
$z + x = 7.$

7. $2x + 3y + 3z = 2,$
$6x - 6y + 5 = 0,$
$y + z = 1.$

8. $x + y + z + 3 = 0,$
$3x + 4y + 5z + 12 = 0,$
$x - 2y - 6z = 7.$

9. $3x + 4y = a,$
$4y + 10z = b,$
$10z + 3x = c.$

10. $A + B + C + D = 0,$
$9A + 5B + 7C + 6D = 2,$
$26A + 19B + 20C + 11D = -28,$
$12A + 6B + 4C + 3D = 0.$

11. $A + B + C = 3,$
$5A + 4B + 3C = 5,$
$6A + 3B + 2C = 2.$

12. $A + C = 1,$
$B + A = 3,$
$2C + B = 1.$

13. $y - x = 2,$
$z - y = 3,$
$x - z = -5.$

14. $x + y + z = 6,$
$x - y + z = 2,$
$x + y - z = 0.$

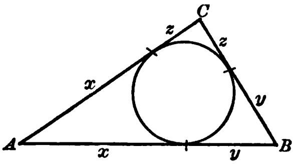
15. $4x + y + 5z + 17 = 0,$
$2x + 2y - 3z = 11,$
$3x + 4y + z = 2.$

16. $3w + 2x - 4y + z = 1,$
$-5w + 4x + 7y + 3z + 9 = 0,$
$w + x + y + z = \frac{1}{2},$
$w - x - y + z = -\frac{9}{2}.$

PROBLEMS

1. Part of the road from A to B is level, part is uphill, and part is downhill. The total distance is 101 miles. A motorcyclist by riding at the rate of 15 miles per hour on the level, 9 miles per hour uphill, and 18 miles per hour downhill made the trip in 8 hours and returned in $7\frac{1}{2}$ hours. How much of the road is level, how much uphill, and how much downhill?

2. It is shown in geometry that the two tangents drawn from a point to a circle are equal. In the accompanying triangle $AB = 7$, $BC = 4$, $CA = 6$. Find x, y, and z.



Circle Inscribed in a Triangle. — A circle is said to be *inscribed* in a triangle when the sides of the triangle are tangent to the circle.

3. If the sides of a triangle are 12, 17, and 10 respectively, where will the inscribed circle touch the sides?

4. Three circles are drawn tangent to each other and with their centers at the vertices of the triangle whose sides are 8, 10, and 12. What are the radii of these circles?

5. The perimeter of a triangle ABC is 21 inches, and the distance from A along AB to the point of contact of the inscribed circle is one third of AB, while along AC it is two thirds of AC. Find the lengths of the sides.

6. The sum of the digits of a certain integer between 100 and 1000 is 10. If the tens' digit and the units' digit are interchanged the number is increased by 18, and if the tens' digit and the hundreds' digit are interchanged, the number is increased by 90. What is the number?

7. The sum of the digits of a certain integer between 100 and 1000 is 6. The units' digit is 1 greater than the tens' digit, which is 1 greater than the hundreds' digit. What is the number?

REVIEW OF CHAPTERS VI AND VII

1. In what sense can an equation in one variable be said to be a description?

2. When is a number said to satisfy such an equation?

3. What is meant by solving an equation?

4. What is a root of an equation?

5. How can you tell whether a given number is a root of a given equation or not?

6. What changes can be made in an equation without affecting the number described by it?

7. Explain the process of clearing an equation of fractions.

8. Give the rule for transposing a term from one member of an equation to the other. What are the reasons for this rule?

9. How many pairs of values of x and y satisfy an equation of the first degree in x and y?

10. How can the information contained in such an equation be exhibited graphically?

11. What are the advantages of this graphical representation?

12. What is meant by the coördinates of a point?

13. What is the equation of the path of a point which moves in such a way that its abscissa is always equal to its ordinate?

14. What is the equation of the path of a point which moves in such a way that the sum of its abscissa and its ordinate is always zero?

15. What is the locus of an equation of the first degree in two variables?

16. Describe three ways of solving a system of two equations of the first degree in two variables.

17. What are independent equations?

18. Give an illustration of a system of equations that are not independent.

19. Give an illustration of inconsistent equations.

20. What is the difference between an identity and an equation of condition?

REVIEW PROBLEMS ON CHAPTERS VI AND VII

1. A marksman heard the sound of his bullet on the target 6 seconds after he fired. How far away is the target? (See Problem 4, page 90.)

2. Three points P, Q, and R on a straight line have the abscissæ -2, -12, and 7 respectively. What must be the abscissa of a fourth point S in order that $\frac{SP}{QP} = 5\frac{SQ}{RQ}$.

3. Where must the fulcrum be placed in order that a uniform rod 6 feet long and weighing 2 pounds to the foot may be balanced by a weight of 3 pounds at one end?

4. Three weights of 2, 5, and 7 pounds respectively are suspended from points 10 inches apart on a straight lever. How far from the point of suspension of the second weight must the fulcrum be placed in order that the weights may be in equilibrium?

5. An automobile starts on a journey at the rate of 16 miles an hour, and 45 minutes later another one starts from the same place and in the same direction at the rate of 22 miles an hour. How long will it be before the second machine overtakes the first one?

6. Two angles of a triangle are each equal to twice the third angle. How many degrees in each angle?

7. What must be the inner radius of a circular ring that is 2 inches wide and contains 63 square inches?

8. Solve Problem 14, § 72, for the case $AB = 6$.

9. Solve Problem 15, § 72, in case the two equal circles have a radius of 4 inches.

10. $12x-(4x-7)=3x-(9x-28)$.

11. Where will the inscribed circle touch the sides of the triangle whose sides are 20 inches, 30 inches, and 26 inches respectively?

12. Two boys balance on a teeter board when one is 6 feet and the other 6 feet 9 inches from the point of support. They find that if the smaller boy holds a 20-pound weight, the larger boy must move out from the point of support 1 foot 6 inches farther in order to preserve the balance. How much does each boy weigh?

13. In the light of Problem 12, describe how two boys could determine their weights by means of a teeter board and a weight.

14. Solve the system of equations

$$\begin{aligned} x+y&=5,\\ y+z&=7,\\ z+x&=3. \end{aligned}$$

15. How must $6000 be distributed between two investments yielding $4\frac{1}{2}\%$ and 5% respectively in order that the total income be $290?

16. The draft power of one horse is two thirds that of another. Where must the doubletree be divided in order that a load pulled by them be justly distributed?

17. Solve the system of equations

$$\begin{aligned} 3x+2y&=10,\\ 2x+3y&=9. \end{aligned}$$

18. Solve the system of equations

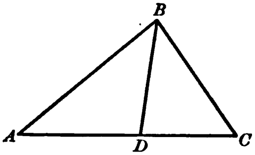
$$\begin{aligned} 5x-2y+3z&=8,\\ -2x+4y-z&=3,\\ x+y-3z&=6. \end{aligned}$$

19. At what temperature will the Fahrenheit thermometer be as many degrees above zero as the Centigrade thermometer is below zero?

20. A man rows 10 miles downstream in $2\frac{1}{4}$ hours and returns in $2\frac{3}{4}$ hours. How fast does he row in still water?

21. Two weights balance on a lever when one is 16 inches and the other is 20 inches from the fulcrum. The balance is maintained if the first weight is increased by 3 pounds and the second weight is moved 4 inches farther away from the fulcrum. What are the weights?

22. If BD bisects the angle ABC and $AB = 10$, $BC = 8$, and $AC = 12$, what are AD and DC?



23. Solve the system of equations

$$y = 3x + 5,$$
$$5x - 6y + 4 = 0.$$

24. A runs around a circular track in 30 seconds, and B in 35 seconds. Two seconds after A starts B starts from the same place in the opposite direction. When will they meet?

25. At what times between 6 and 7 o'clock are the hands of a watch at right angles to each other?

26. If the first machine in Problem 5 is delayed for 10 minutes a quarter of an hour after it starts, when and where will the second machine overtake it?

27. Solve Problem 26 graphically.

28. Divide the number n into two parts one of which is four times the other.

29. The excess of a number over a is equal to three times its excess over b. What is the number?

30. A is m, and B is n, years old. In how many years will A be twice as old as B?

31. How far can a person who has 3 hours to spare ride in a machine at the rate of 20 miles an hour so that he can return in a carriage at the rate of 7 miles an hour?

32. How long between conjunctions of the earth and Jupiter? (See Problems 20 and 21, p. 92.)

33. A rectangular plot of ground four times as long as it is broad is to be laid out with a path 4 feet wide running around it. What must be the length of the plot in order that the area of the path be 4064 square feet?

34. A bar 12 feet long weighing 3 pounds to the foot has a weight of 8 pounds attached to one end. How far from this end must the point of support be in order that the bar may balance?

35. How long will it take $860 at 5%, simple interest, to amount to $1032?

36. At what rate of interest will $925 amount to $1295 in 10 years at simple interest?

37. Solve the system of equations

$$y-1=3(x-2),$$
$$y-1=5(x-2).$$

38. The tens' digit of a certain integer between 10 and 100 is 5 more than the units' digit, and if the digits were reversed the number would be decreased by 45. What is the number?

39. Determine graphically how long it will take a bicyclist going at the rate of 12 miles an hour to overtake a pedestrian going at the rate of 4 miles an hour, if the latter has a start of 6 miles.

40. If the sides of a triangle are 9, 12, and 15 respectively, where will the inscribed circle touch the sides?

CHAPTER VIII

DIVISION OF POLYNOMIALS

88. The student has already learned how to divide monomials and polynomials by a monomial. We come now to the more difficult task of dividing one polynomial by another polynomial.

In dividing a polynomial A by a polynomial B we seek a polynomial Q such that

$$A \equiv BQ.$$

We can therefore learn how to perform this division by a study of the multiplication of polynomials by polynomials. Consider the product of the two factors $4x^3 - 5x^2 - 7x + 1$ and $x + 4$, which are arranged according to the descending powers of x.

$$\begin{array}{rrrrr}
x & +\ 4 & & & \\
4x^3 & -\ 5x^2 & -\ 7x & +1 & \\ \cline{1-4}
4x^4 & +\ 16x^3 & & & \\
 & -\ 5x^3 & -\ 20x^2 & & \\
 & & -\ 7x^2 & -28x & \\
 & & & +x & +4 \\ \hline
4x^4 & +\ 11x^3 & -\ 27x^2 & -27x & +4
\end{array}$$

NOTE.—This is not the most compact way to arrange this work for finding the product of these two factors, but it is the best way for our present purpose.

Suppose now that we have given the product $4x^4 + 11x^3 - 27x^2 - 27x + 4$ and the factor $x + 4$, and wish to find the other factor.

$$\begin{array}{rrrrrl}
4x^4 & +\ 11x^3 & -\ 27x^2 & -27x & +4 & \multicolumn{1}{|l}{x+4} \\ \cline{6-6}
4x^4 & +\ 16x^3 & & & & 4x^3 - 5x^2 - 7x + 1 \\ \cline{1-2}
 & -\ 5x^3 & -\ 27x^2 & & & \qquad \text{Quotient} \\
 & -\ 5x^3 & -\ 20x^2 & & & \\ \cline{2-3}
 & & -\ 7x^2 & -27x & & \\
 & & -\ 7x^2 & -28x & & \\ \cline{3-5}
 & & & x & +4 & \\
 & & & x & +4 & \\ \cline{4-5}
 & & & & 0 &
\end{array}$$

Since the first term of the product (dividend) is equal to the product of the first terms of the factors, the first term of the quotient (which is one of the factors) is obtained by dividing the first term of the dividend by the first term of the divisor. If we multiply the divisor by this first term of the quotient the product is one of the partial products obtained in multiplying the two factors (divisor and quotient) together. Now the total product is the sum of such partial products. We therefore subtract this first partial product from the dividend, and the remainder is the product of the divisor and the part of the quotient not yet found. Hence the second term of the quotient will be obtained by dividing the first term of this remainder by the first term of the divisor. If now we multiply the divisor by this second term of the quotient, the result will be the second partial product obtained in the multiplication of the divisor and the quotient. If we subtract this partial product from the preceding difference, the third term of the quotient is obtained by dividing the first term of the resulting difference by the first term of the divisor. If we continue this process until we get a final difference equal to zero, it is evident that the product of the divisor and the quotient is the dividend.

Check. — If we substitute numerical values for the letters, the value of the quotient ought to equal the quotient of the values of the dividend and of the divisor. This fact may be used to check the accuracy of the work of division, or we can use the fact that the dividend should be the product of the divisor and the quotient.

89. If there is no polynomial Q such that $A \equiv BQ$, we can arrange A and B in descending powers of some common letter and continue the formal work of division until a remainder is obtained that is of lower degree in this letter than the divisor. If we denote this remainder by R, we have

$$A \equiv BQ + R.$$

Or, we can arrange A and B in ascending powers of some common letter and continue the formal work of division until the degree in this letter of the next term of the quotient would be greater than the difference of the degrees of A and B. If we denote the resulting remainder by R, we have, as before,

$$A \equiv BQ + R.$$

Definition. — The process of finding Q and R is called the process of *dividing A by B.*

90. The steps to be taken in dividing one polynomial by another are described in the following

Rule. — 1. *Arrange the dividend and the divisor according to the descending (or ascending) powers of some common letter, called the letter of arrangement.*

2. *Divide the first term of the dividend by the first term of the divisor. Take this quotient as the first term of the quotient sought.*

3. *Multiply the divisor by the first term of the quotient and subtract the product from the dividend.*

4. *Divide the first term of the remainder by the first term of the divisor and take the quotient as the second term of the quotient sought.*

5. *Proceed with this second term of the quotient as with the first term, and continue the process until a zero remainder or one of lower degree than the divisor in the letter of arrangement is obtained, in case the dividend and the divisor have been arranged in descending powers of the letter of arrangement. If the arrangement is in ascending powers, the process should be continued until there is obtained a zero remainder, or until the degree of the next term of the quotient would be greater than the difference of the degrees of the dividend and the divisor.*

If a zero remainder is reached the division is said to be *exact*, and the dividend is identically equal to the product of the quotient and the divisor.

$$\text{Dividend} \equiv \text{Quotient} \times \text{Divisor}.$$

If a zero remainder is not reached,

$$\text{Dividend} \equiv \text{Quotient} \times \text{Divisor} + \text{Remainder}.$$

We have a case analogous to this in arithmetic. For example, if we divide 23 by 5, the quotient is 4 and the remainder is 3, and this may be expressed by saying

$$23 = 4 \times 5 + 3.$$

We add an example of the last case referred to in the rule.

Example.

$$
\begin{array}{rrrrr|l}
3 + & 2x - & 3x^2 + & 5x^3 + & x^4 & \underline{1 + 4x - 2x^2} \\
3 + & 12x - & 6x^2 & & & 3 - 10x + 43x^2 \\
\hline
 & -10x + & 3x^2 + & 5x^3 & & \\
 & -10x - & 40x^2 + & 20x^3 & & \\
\hline
 & & 43x^2 - & 15x^3 + & x^4 & \\
 & & 43x^2 + & 172x^3 - & 86x^4 & \\
\hline
 & & & -187x^3 + & 87x^4 &
\end{array}
$$

Here the degree of the next term of the quotient would be 3, and the difference of the degrees of the dividend and the divisor is 2. We have then $3 - 10x + 43x^2$ as a quotient and $-187x^3 + 87x^4$ as a remainder.

To check the accuracy of the division when there is a remainder, multiply the quotient by the divisor and add the remainder to the product. The result should be the dividend.

91. If we had arranged the dividend and the divisor according to the descending powers of x, we would have obtained $-\frac{1}{2}x^2 - \frac{7}{2}x - \frac{23}{4}$ and $\frac{57}{2}x + \frac{35}{4}$ for quotient and remainder respectively. That is,

$$x^4 + 5x^3 - 3x^2 + 2x + 3 = (-\tfrac{1}{2}x^2 - \tfrac{7}{2}x - \tfrac{23}{4})(-2x^2 + 4x + 1) + \tfrac{57}{2}x + \tfrac{35}{4}.$$

The student will recall from his arithmetic that when one number is divided by another only one result can be obtained. The example just given shows that this is not always the case in the division of one polynomial by another polynomial. If, however, the division is exact for one arrangement (ascending or descending) of the letter of the arrangement, it will be exact for the other arrangement (descending or ascending) and the two quotients will be the same.

EXERCISES

Divide:

1. $x^2 + 5x + 6$ by $x + 2$.
2. $x^2 - 6x + 5$ by $x - 5$.
3. $y^2 - 8y + 15$ by $y - 3$.
4. $a^2 - 2a - 35$ by $a - 7$.
5. $a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$ by $a + b$.
6. $x^3 + 9x^2 + 27x + 27$ by $x + 3$.
7. $x^3 + 6x^2 - 5x - 50$ by $x + 5$.

8. $3x^4 - x^3 - 5x^2 + 2x - 32$ by $x + 2$.

9. $a^3 + 2a^2b + 2ab^2 + b^3$ by $a + b$.

10. $x^4 - 13x^2 + 36$ by $x - 2$.

11. $x^3 - 8x^2 + 17x + 26$ by $x + 1$.

12. $1 - 3x^2 + 9x^4$ by $1 - 3x + 3x^2$.

13. $3m^5 + 17m^4 + 20m^3 + 25m^2 + 13m + 2$ by $m^2 + 5m + 2$.

14. $6b - 4 + 5b^3 - b^5 - 2b^4$ by $2 + b^2 - b^3$.

15. $6x^3 - 8xy^2 + 7x^2y - 5y^3$ by $3x^2 - 5y^2 + 2xy$.

16. $3a^2b^2 - 2a^3b + b^4 + a^4 - 2ab^3$ by $a^2 + b^2 - ab$.

17. $a^3 + b^3$ by $a + b$.

18. $a^3 - 64c^3$ by $a - 4c$.

19. $125x^3 + 64y^3$ by $5x + 4y$.

20. $m^4 - n^4$ by $m - n$.

21. $m^4 - n^4$ by $m + n$.

22. $a^4 + x^4$ by $a + x$.

23. $a^5 + b^5$ by $a + b$.

24. $a^5 - b^5$ by $a - b$.

25. $b^5 - 32$ by $b + 2$.

26. $x^5 - y^5$ by $x + y$.

27. $x^5 + y^5$ by $x - y$.

28. $m^4 + n^4$ by $m + n$.

29. $x^4 + x^2y^2 + y^4$ by $x^2 + xy + y^2$.

30. $x^6 - y^6$ by $x - y$.

31. $x^6 - y^6$ by $x + y$.

32. $x^6 + y^6$ by $x - y$.

33. $x^6 + y^6$ by $x + y$.

34. $x^6 + y^6$ by $x^2 + y^2$.

35. $x^6 + y^6$ by $x^2 - y^2$.

36. $x^6 - y^6$ by $x^2 + y^2$.

37. $x^4 + 12x^3 + 54x^2 + 108x + 81$ by $x^3 + 9x^2 + 27x + 27$.

38. $a^{3n} + 9a^{2n} + 26a^n + 24$ by $a^n + 3$.

39. $a^{3x} + 3a^{2x}b^y + 3a^xb^{2y} + b^{3y}$ by $a^x + b^y$.

40. $x^{a+2} + 6x^{a+1} + 5x^a + 3x^2 + 18x + 15$ by $x^2 + 6x + 5$.

CHAPTER IX

PRODUCTS AND FACTORS

92. Powers of Monomials. — We have seen (§ 46) how to form the product of two or more monomials. If these monomials are all the same the product is called a *power* of the monomial. The number of times the monomial is taken as a factor is called the *index* of the power.

Thus the product $2\,a^2b \cdot 2\,a^2b \cdot 2\,a^2b$ is the third power of $2\,a^2b$ and 3 is the index of the power. Instead of writing all the factors, it is customary and desirable to write one of them with an exponent equal to the index of the power. Thus the product just considered is written $(2\,a^2b)^3$.

It is clear that this product equals $8\,a^6b^3$, and the general rule for forming a positive integral power of a monomial is as follows:

Rule. — *A power of a monomial is a monomial in which the numerical coefficient is the numerical coefficient of the monomial raised to the given power and in which the exponent of any letter is the product of the exponent of this letter in the monomial by the index of the power.*

The area of a square is equal to the second power of the length of a side, and the volume of a cube is equal to the third power of the length of an edge. Hence the second power of a number is called its *square*, and the third power its *cube*.

EXERCISES

By the values of the following expressions we mean the equivalent expressions obtained by the application of the preceding rule.

Give at sight the values of the following:

1. 4^2.
2. 7^3.
3. 3^5.
4. 1^5.
5. $(-6)^2$.
6. $(-\frac{5}{2})^2$.
7. $(7\frac{1}{2})^2$.
8. $(-1)^3$.
9. $(-1)^4$.
10. $(-9)^2$.
11. $(\frac{4}{3})^2$.
12. $(-5)^3$.
13. $(\frac{8}{3})^3$.
14. $(-2)^5$.
15. $(-1)^5$.
16. $(lmn)^3$.
17. a^4.
18. $(2\,a)^4$.
19. $(a^2)^4$.
20. $(-2\,a)^4$.
21. $(-a^2)^4$.
22. $(2\,a^2b^2)^2$.
23. $(-x)^5$.
24. $(-xy)^5$.
25. $(-2\,x)^5$.
26. $(-2\,x^2)^5$.
27. $(2\,xy^2)^7$.
28. $(-xy^2)^7$.
29. $(3\,abxy)^7$.
30. $(3\,a^2x^3)^2$.
31. $(\frac{5}{2}\,a^2b^4)^3$.
32. $(2\frac{1}{2}\,by)^2$.
33. $(-\frac{4}{8}\,b^3c^3)^2$.
34. $(\frac{2}{5}\,abx)^2$.
35. $(-6\,ab)^3$.
36. $(3\,a^2x)^3$.
37. $(\frac{5}{6}\,xy)^3$.
38. $(-ab^3)^3$.
39. $(2\,x^3y^2)^5$.
40. $(-3\,a^2bm)^5$.
41. $(-\frac{2}{8}\,b^4c^3)^5$.
42. $(\frac{10}{3}\,a^4b^2x^5y^3)^2$.
43. $(-3\,a^3b^2)^3$.
44. $(-2\,a^2b^2)^2$.
45. $(\frac{5}{6}\,a^3b^4)^3$.
46. $(-\frac{3}{4}\,x^3y^2z^4)^3$.
47. $(-.3\,a^4x^5)^2$.
48. $(-..3\,a^4x^5)^4$.
49. $(-3\,a^4b^5)^5$.
50. $(\frac{2}{5}\,a^4b^2)^3$.
51. $(-\frac{2}{5}\,a^4b^2)^3$.
52. $(-xyz)^6$.
53. $-(xyz)^6$.
54. $(-3\,by^3)^1$.
55. Show that $(4\,m^2n^3)^6 = [(4\,m^2n^3)^3]^2$.
56. Show that $(4\,m^2n^3)^6 = [(4\,m^2n^3)^2]^3$.
57. Show that $(a^xb^y)^4 = [(a^xb^y)^2]^2$.

93. Monomial Factors of Polynomials.—The method of writing a monomial is such that all its literal factors are clearly exhibited. If the numerical coefficient of the monomial is an integer, the numerical factors of the monomial are the factors of this coefficient. But if this numerical coefficient is not an integer, we shall say that there are no numerical factors of the monomial except its numerical coefficient and 1; although in a broader sense we might say that any number is a numerical factor of the monomial.

Since the product of a polynomial by a monomial is a polynomial whose terms are the terms of the given polynomial multiplied by the monomial, it follows that any monomial factor of a polynomial must be a factor of each term of the polynomial, and that any common factor of the terms of a polynomial is a factor of this polynomial.

For example, the polynomial $3x^2 + 6xy$ contains the monomial factor $3x$, and we have

$$3x^2 + 6xy = 3x(x + 2y).$$

This method is also applicable in case none of the terms of the polynomial contains any literal factors. Thus,

$$2 - 8 + 4 + 100 = 2(1 - 4 + 2 + 50).$$

When we have expressed a polynomial as the product of certain factors, we can check our result by multiplying these factors together. The product should, of course, equal the original polynomial.

EXERCISES

Write the following polynomials as products of monomials and polynomials:

1. $5 + 20 - 35 + 70.$
2. $4 \cdot 6 + 2 \cdot 3 - 12.$
3. $3 \cdot 4 \cdot 5 - 4 \cdot 5 \cdot 6 + 7 \cdot 12.$
4. $2 \cdot 3 \cdot 4 \cdot 5 \cdot 8 - 4^2 \cdot 7 \cdot 8.$
5. $2^3 \cdot 3^4 \cdot 5 - 2 \cdot 3^3 \cdot 5^4 - 2^2 \cdot 3^4 \cdot 5^2.$
6. $7^2 \cdot 11 \cdot 15 + 5^3 \cdot 3^2 \cdot 22 + 7 \cdot 5 \cdot 3.$
7. $21\,a^3b^2 - 3\,a^2b^3 + 6\,b^4.$
8. $4\,rst - 8\,r^3t^3 + 2\,rt^2u.$

9. $ay^2 + aby + y^3$.

10. $p^7 + pq^6$.

11. $4\,m^3x^3 - 8\,m^5x^5 + 40\,m^2x^2$.

12. $7\,a^2 - 6\,a^3 + a^4 - 4\,a$.

13. $abc + ax^2 + abx + acx$.

14. $2\,x^5 + 14\,x^4 + 24\,x^3$.

15. $4\,xy^2 + 8\,x^3 + 8\,x^2y$.

16. $6\,x^4y + 18\,x^2y - 12\,xy + 15\,x^3y$.

17. $7\,x^4y^2z^2 + 4\,x^3y^3z^2 - 5\,x^2y^3z^3$.

18. $a^3 - 2\,a^2b + ab^2$.

19. $3\,m^4n - 9\,m^3n^2 + 9\,m^2n^3 - 3\,mn^4$.

20. $3\,a^4b^2 + 6\,a^3b^3 - 12\,a^6 + 9\,b^6$.

21. $10\,xy - 15\,y^3 - 35\,x^5$.

22. $48\,x^5y^2 + 288\,x^4y^3 + 648\,x^3y^4 + 162\,x^2y^5 + 243\,xy^6$.

23. $4\,x^3 + 4\,x - 100$.

24. $x^4 + 3\,x^3y + 3\,x^2y^2 + xy^3$.

25. $3\,x^5 - 12\,x^3y^2 + 6\,xy^4$.

26. $9\,b^4m^3n^2 + 27\,b^3m^4n^3 + 45\,b^2m^5n^4$.

27. $16\,r^2s^2x - 24\,rsx^3 + 32\,r^4s^4x^4$.

28. $-16\,r^2s^2x + 24\,rsx^3 - 32\,r^4s^4x^4$.

29. $10 \cdot 2^3 \cdot 3^3 - 7 \cdot 2^2 \cdot 3^4 + 13 \cdot 2 \cdot 3^5$.

30. $a^{x+2} + a^{x+1}b + a^xb^2$.

31. $2\,a^mb^n + 4\,a^{m+1}b^n + 6\,a^mb^{n+2}$.

32. $x^{2m}y^n + x^my^{2n} + x^my^n$.

33. $4\,a^xb^y + 10\,a^{3x}b^{3y} - 32\,a^{5x}b^{2y}$.

34. $7\,a^{3m+2}x^{3n+5} - 12\,a^{2m+1}x^{n+4} + 3\,a^{m+1}x^{n+2}$.

94. Factors found by grouping Terms. — Suppose we wish to find the factors of $ax + 2\,y + ay + 2\,x$. We can rewrite this polynomial thus,

$$\begin{aligned}&(ax + ay) + (2\,x + 2\,y)\\ &= a(x + y) + 2(x + y)\\ &= (x + y)(a + 2).\end{aligned}$$

This example suggests that we proceed, where possible, in accordance with the following:

Rule. — *Arrange the terms of the polynomial to be factored in groups of two or more terms each, such that the polynomial formed by the terms of each group is the product of a monomial and a polynomial, this polynomial factor being the same for each group. The original polynomial is then the product of this polynomial factor and a polynomial whose terms are the respective monomial factors.*

EXERCISES

Factor the following polynomials:

1. $4(a+b)-x(a+b)$. 2. $a(x-y)-3\,b(x-y)$.
3. $7\,a(2\,x-3\,y)+2\,b(2\,x-3\,y)$
4. $lmn(3\,a+5\,b)-xy(3\,a+5\,b)$.
5. $3\,x(a+b+c)+2\,y(a+b+c)$
6. $a^2(a+2\,b-4\,c)-ab(a+2\,b-4\,c)+b^2(a+2\,b-4\,c)$.
7. $4\,a^2(l-3\,m)-5\,b^2(l-3\,m)+2\,c^2(l-3\,m)$.
8. $5\,x^2y^2z^2(a^2+2\,ab+b^2)-4\,xyz(a^2+2\,ab+b^2)$
$+3(a^2+2\,ab+b^2)$.
9. $ax+cx+ay+cy$.
10. $3\,al+3\,bl+2\,am+2\,bm$.
11. $3\,ax+2\,ay+2\,by+3\,bx+3\,cx+2\,cy$.
12. $3+7\,x-9\,c-21\,cx$.
13. $1+a+a^2-x-ax-a^2x$.
14. $a^3x-b^3x-b^3y-b^3z+a^3z+a^3y$.
15. $2\,x-3\,y+9\,aby-6\,abx$.
16. $ax+ay+bx+bz+az+by+cz+cx+cy$.
17. $4\,a(7\,l-m)-5(m-7\,l)+8\,b(7\,l-m)$.
18. $4\,x^3+8\,x-4\,x^2-8$.
19. $a^2x^2+b^2y^2-b^2x^2-a^2y^2$.
20. $6\,a^2bx-8\,abcx-15\,ab^2y+20\,b^2cy$.

21. $ax - bx - ay + by$.

22. $ax + bx + ay + az + by + bz$.

23. $3\,mx + 2\,my + 3\,nx + 2\,ny$.

24. $x(a - b) + 2\,y(b - a)$.

25. $m(x - 3\,y) - 3(2\,x - 6\,y)$.

26. $a^3 + 2\,a^2 + a + 2$.

27. $b^3 - 3\,b^2 + 2\,b - 6$.

28. $x^4 - x^3 + 2\,x - 2$.

29. $2\,a^3 + 4\,a^2 + 3\,a + 6$.

30. $a(3\,x - 4\,y) - 4\,b(4\,y - 3\,x)$.

31. $10\,a^2 - 8\,ab - 15\,ab^2 + 12\,b^3$.

32. $al + am + an + bl + bm + bn$.

33. $a^3x^3 + a^3 - x^3 - 1$.

34. $r^3 - r^2 + 3\,r - 3$.

35. $2\,lr - 5\,ms - mr + 10\,ls$.

36. $2\,b^3 + 7\,b^2c + 2\,bc^2 + 7\,c^3$.

37. $a^4 - ab - a^3b^3 + b^4$.

38. $a^4 + ab + a^3b^3 + b^4$.

39. $ab + ay + bx + xy$.

40. $abxy + 3\,by + 2\,ax + 6$.

95. The Square of a Binomial. — The student should verify that the product of $a + b$ and $a + b$ is $a^2 + 2\,ab + b^2$.

$$(a + b)^2 = a^2 + 2\,ab + b^2.$$

This can be expressed in ordinary language as follows:

The square of the sum of two terms is equal to the square of the first term plus twice the product of the first by the second, plus the square of the second term.

In a similar way the student should form the product of $a - b$ and $a - b$ and formulate the result in ordinary language.

The polynomial $a^2 + 2\,ab + b^2$ is called the *expansion* of $(a + b)^2$ because it is a longer expression for the same thing. But, although it is the longer and the more cumbersome of the two, it is, for many purposes, the more useful form. Similar remarks apply to other powers of binomials. For this reason the student should become proficient in the expansion of powers of binomials. (See § 97.)

These results can sometimes be used to advantage in finding the square of a number by first writing the given number as the sum or difference of two numbers.

Thus, $\overline{21}^2 = (20 + 1)^2 = 20^2 + 2 \cdot 20 \cdot 1 + 1^2 = 441$. Any number can be written as the sum or difference of two numbers in many different ways. We might have said, for example, $\overline{21}^2 = (13 + 8)^2 = 13^2 + 2 \cdot 13 \cdot 8 + 8^2 = 169 + 208 + 64 = 441$. The result is, of course, the same as before; but in the former case it was easier to perform the necessary operations. In finding the square of a number by this method, the student should therefore be careful to make a suitable choice of the numbers whose sum or difference equals the given number.

EXERCISES

Find the squares of the following:

1. 99.
2. 101.
3. 999.
4. 1001.
5. 51.
6. 35.
7. $m + n$.
8. $m - n$.
9. $m + 2n$.
10. $3m + n$.
11. $3m + 2n$.
12. $x^2 - 5$.
13. $a^2 + 4c$.
14. $8x^3 + 4y^2$.
15. $-5m^2 + 2n^6$.
16. $-8a^2 - 5x^3$.
17. $-12a^2b - 11x^3y$.
18. $\frac{4}{3}abx + \frac{2}{5}bcy$.
19. $a^4b^4 - b^4y^2$.
20. $.5a^3b^2 - .42abx^2$.
21. $6m^4n^5 - 5m^4n^3$.
22. $3xyz + 2$.
23. $-.7a^3 - .11b^3$.
24. $2m^2 - 6m^3y$.
25. $7abc - 4xyz$.

With these two rules in mind an indicated division can sometimes be performed by inspection.

For example, when we recognize that $a^2 + 2ax + x^2$ is the square of $a + x$, we can say immediately that when we divide the former by the latter, the quotient is $a + x$; that is, $\frac{a^2 + 2ax + x^2}{a + x} = a + x$.

Perform the following indicated divisions by inspection:

26. $\frac{a^2+2ab+b^2}{a+b}$.

27. $\frac{a^2+6a+9}{a+3}$.

28. $\frac{a^2-6a+9}{a-3}$.

29. $\frac{x^2+2xy+y^2}{x+y}$.

30. $\frac{x^2-4xy+4y^2}{x-2y}$.

31. $\frac{x^2-4xy+4y^2}{2y-x}$.

32. $\frac{25-10b+b^2}{b-5}$.

33. $\frac{m^2+2mn+n^2}{m+n}$.

34. $\frac{m^2-2mn+n^2}{m-n}$.

35. $\frac{a^2-6ab+9b^2}{a-3b}$.

36. $\frac{4x^2+12xy+9y^2}{-2x-3y}$.

37. $\frac{25a^2-40ab+16b^2}{5a-4b}$.

38. $\frac{49a^2b^2c^2+56abcxy+16x^2y^2}{7abc+4xy}$.

39. $\frac{9x^2-30x+25}{3x-5}$.

40. $\frac{100a^4b^6+100a^2b^3x^2+25x^4}{10a^2b^3+5x^2}$.

96. Since the square of a binomial is a trinomial of a certain form, we can determine by inspection whether or not a given trinomial is the square of some binomial and, if it is, what that binomial is.

For example, $9a^2-24ab+16b^2=(3a-4b)(3a-4b)=(3a-4b)^2$. If we change the signs of all the terms in both factors, the product will not be affected. Hence, also $9a^2-24ab+16b^2=(4b-3a)^2$.

Definition. — A trinomial that is the square of a binomial is called a *trinomial square.*

EXERCISES

1. Formulate a rule by means of which you can determine whether any polynomial is the square of a binomial.

2. Tell how the binomial can be found when you know that the polynomial is the square of a binomial.

Apply these rules to finding the equal binomial factors of those of the following polynomials that have such factors:

3. $x^2 + 4\,xy + 4\,y^2$.
4. $x^2 + 4\,y^2 - 4\,xy$.
5. $a^2 - ab + b^2$.
6. $4\,m^2 + 9\,n^2 - 12\,mn$.
7. $a^2b^4c^6 + 25\,b^4c^8 - 10\,ab^4c^7$.
8. $x^2 - xy + y^2$.
9. $36 + 49 - 84$.
10. $x^4 + x^2y^2 + y^4$.
11. $4\,a^2 + 4\,y^2 - 8\,ay$.
12. $a^2 + 2\,ab + 4\,b^2$.
13. $a^2 + 2\,ab - b^2$.
14. $9\,a^2 + 24\,ax + 16\,x^2$.
15. $-x^2 - 2\,xy - y^2$.
16. $9\,m^4 + 4\,n^2 - 12\,mn$.
17. $81\,b^2 - 108\,bc + 36\,c^2$.
18. $36\,b^2 - 108\,bc + 81\,c^2$.
19. $k^2 + 9\,l^2 - 6\,lk$.
20. $24\,mn + 36\,n^2 + 4\,m^2$.
21. $4\,a^2 + 4\,ab - b^2$.
22. $-36\,x^2 - 36\,xy + 9\,y^2$.
23. $k^4 + 9 - 6\,k^2$.
24. $m^6n^6 + a^8b^8 + 2\,a^4b^4m^3n^3$.
25. $9\,a^2 - 12\,ay + 4\,y^2$.
26. $9\,a^2 + 12\,ay - 4\,y^2$.
27. $12\,ay - 9\,a^2 - 4\,y^2$.
28. $-10\,x^2 + 25 + x^4$.
29. $a^4 + 2\,a^2b^2 + b^4$.
30. $x^6 - 2\,x^3y^3 + y^6$.
31. $x^9 - 2\,x^3y^3 + y^9$.
32. $(a + b)^2 + 2\,c(a + b) + c^2$. Consider $(a + b)$ as one term.
33. $4(x + 2)^2 + 4(x + 2) + 1$.
34. $9(a - b)^2 + 4(x + y)^2 - 12(a - b)(x + y)$.
35. $(a + c)^2 + 2\,x(a + c) + x^2$.
36. $9(x + y)^2 + 3(x + y) + 9$.
37. $a^{2x} + 2\,a^x + 1$.
38. $a^{2x} + 2\,a^xb^y + b^{2y}$.
39. $9\,x^{2m} + 30\,x^my^n + 25\,y^{2n}$.
40. $(x + 1)^2 - 2(x + 1)(y - 1) + (y - 1)^2$.

97. The Cube of a Binomial. — The student should verify by multiplication the following two formulæ:

$$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3,$$
$$(x-y)^3 = x^3 - 3x^2y + 3xy^2 - y^3.$$

He should also state these formulæ in ordinary language.

EXERCISES

Expand the following:

1. $(a+b)^3$.
2. $(a-b)^3$.
3. $(a+1)^3$.
4. $(b+2)^3$.
5. $(x+3)^3$.
6. $(3a+b)^3$.
7. $(4k-3l)^3$.
8. $(100-1)^3$.
9. $(50+1)^3$.
10. $(3x^2-5y^3)^3$.
11. $(a+4)^3$.
12. $(m-1)^3$.
13. $(5+y)^3$.
14. $(4ab-3xy)^3$.
15. $(2x^2y-5k^3l^4)^3$.
16. $(7ab-4bc)^3$.
17. $(2x+3y)^3$.
18. $(a^3+b^3)^3$.
19. $[(a+b)+c]^3$.
20. $[x-(y+z)]^3$.
21. $[(x+y)-4]^3$.

22. Formulate a rule by means of which you can determine whether a given polynomial is the cube of a binomial.

23. Tell how the binomial can be found when you know that the polynomial is the cube of the binomial.

Apply these rules to finding the equal binomial factors of the following polynomials:

24. a^3+3a^2+3a+1.
25. $b^3+6b^2+12b+8$.
26. y^3-3y^2+3y-1.
27. $a^3+3a^2x+3ax^2+x^3$.
28. $8a^3+12a^2b+6ab^2+b^3$.
29. $8a^3-12a^2b+6ab^2-b^3$.

30. b^3+3b^2+3b+1.

31. $m^3-1+3m-3m^2$.

32. $8a^3-36a^2b+54ab^2-27b^3$.

33. $27x^3-108x^2y+144xy^2-64y^3$.

34. $x^3-9x^2y+27xy^2-27y^3$.

35. $27m^3+54m^2+36m+8$.

36. $x^{3m}+3x^{2m}+3x^m+1$.

37. $a^{3x}-3a^{2x}b^y+3a^xb^{2y}-b^{3y}$.

38. $(a+b)^3+3(a+b)^2+3(a+b)+1$.

39. $(x+y)^3-3(x+y)^2+3(x+y)-1$.

98. The Product of the Sum and the Difference of Two Terms. — The student should verify by multiplication that

$$(x+y)(x-y)=x^2-y^2,$$

and should write out this formula in ordinary language.

EXERCISES

Expand the following by means of this formula:

1. $(5+2)(5-2)$.

2. $(9-5)(9+5)$.

3. $(10+3)(10-3)$.

4. $(12-2)(12+2)$.

5. $12\cdot 18$. Write this as $(15-3)(15+3)$.

6. $18\cdot 32$.

7. $24\cdot 28$.

8. $56\cdot 64$.

9. $91\cdot 99$.

10. $(a+b)(a-b)$.

11. $(k-3l)(k+3l)$.

12. $(-7x+2y)(7x+2y)$.

13. $(4+x)(4-x)$.

14. $(a+7)(a-7)$.

15. $(m+3n)(m-3n)$.

16. $(3a+5b)(3a-5b)$.

17. $(9d-4e)(9d+4e)$.

18. $(x^2+y^2)(x^2-y^2)$.

19. $(k^3l^2-xy^4)(k^3l^2+xy^4)$.

20. $(-5m^3+n^3)(5m^3+n^3)$.

21. $(a^4b^2+b^3c)(b^3c-a^4b^2)$.

22. $(-5x^4y^2 + 4x^3y)(4x^3y + 5x^4y^2)$.

23. $(x^6 + y^5)(x^6 - y^5)$. 24. $(abxy - xy)(xy + abxy)$.

25. $(-4k^2 - 5l^2)(4k^2 - 5l^2)$.

26. $(3ab^3 - 2a^3b)(-3ab^3 - 2a^3b)$.

27. $(2x^4y^2 - 3x^2y^4)(-2x^4y^2 - 3x^2y^4)$.

If in the formula we put $a+b$ in place of x, we get

$$(a+b+y)(a+b-y) = (a+b)^2 - y^2 = a^2 + 2ab + b^2 - y^2.$$

If we put $c+d$ in place of y, we get

$$(x+c+d)(x-c-d) = x^2 - (c+d)^2 = x^2 - c^2 - 2cd - d^2.$$

By using these formulæ, perform the following indicated multiplications:

28. $(x+y+2)(x+y-2)$. 29. $(x-y-4)(x-y+4)$.

30. $(3a-2b+c)(3a-2b-c)$.

31. $[(a-b)+2y][(a-b)-2y]$.

32. $[a+(x+y)][a-(x+y)]$.

33. $[(2x+3y)+4z][(2x+3y)-4z]$.

34. $(2a+3b+4c)(2a-3b+4c)$.

35. $[a-(b-c)][a-(b+c)]$.

36. $(x+3y-5z)(x-3y+5z)$.

Perform at sight the following indicated divisions:

37. $\dfrac{x^2-y^2}{x-y}$.

38. $\dfrac{a^2-9b^2}{a+3b}$.

39. $\dfrac{36m^2-49n^2}{6m+7n}$.

40. $\dfrac{k^4-1}{k^2-1}$.

41. $\dfrac{a^6-b^6}{a^3+b^3}$.

42. $\dfrac{x^6-9y^6}{x^3-3y^3}$.

43. $\dfrac{m^{10}-16n^{10}}{m^5-4n^5}$.

44. $\dfrac{81x^4-36y^6}{9x^2-6y^3}$.

99. When a polynomial has been factored into the product of two or more polynomials (or monomials), it may be that some of the factors can be factored into the product of other factors. When a polynomial is to be factored, the factoring should be carried as far as possible.

Definition. — A polynomial that cannot be factored into the product of two polynomials is said to be *prime*.

Thus $a - b$ is prime, while $a^2 - b^2$ is not prime. We use the term "polynomial" here in the broad sense to include monomials.

Since in the definition of a polynomial we have not restricted the coefficients to being integers, every polynomial is divisible by every number. Thus, $a - b$ is divisible by 4 and the quotient is $\frac{1}{4} a - \frac{1}{4} b$. But usually in discussing polynomials we are thinking of polynomials with integral coefficients, and accordingly we shall say that 4 is not a factor of $a - b$. In general, we shall say that an integer is a factor of a polynomial with integral coefficients if it is a divisor of the numerical coefficients of the polynomial, and that a polynomial with integral coefficients is a factor of a given polynomial with integral coefficients if the quotient of the latter by the former has integral coefficients.

When a polynomial that is to be factored contains a monomial factor, this should be removed first and then the other factor should be factored as far as possible.

100. The Factors of the Difference of Two Squares. — We have seen that the product of the sum and the difference of two numbers is equal to the difference of the squares of these numbers. Conversely, the difference of the squares of any two numbers is equal to the product of the sum and the difference of these numbers. We can make use of this fact to find the factors of polynomials that can be expressed as the difference of the squares of two polynomials.

EXERCISES

Find the factors of the following:

1. $x^2 - y^2$.
2. $16x^2 - 9$.
3. $a^2 - 9b^2$.
4. $4c^2 - 25d^2$.
5. $b^4 - y^2$.
6. $36 - x^2$.
7. $x - x^3$.
8. $m^4n^2 - 49$.
9. $a^3 - ab^2$.
10. $x^5 - xy^4$.
11. $a^6 - b^6$.
12. $m^{16} - n^{16}$.
13. $5x^2 - 5y^2$.
14. $7a^2x - 7b^2x$.
15. $49x^8 - 64y^8$.
16. $64k^2 - 289$.
17. $a^2 - b^{2x}$.
18. $121a^4 - 64b^2$.
19. $3a^2xy - 27b^4x^3y^3$.
20. $x^{2n} - y^{2n}$.
21. $(x + y)^2 - 9$.
22. $(x - y)^2 - 16a^2$.
23. $c^2 + 2cd + d^2 - k^2$. *Suggestion.* $c^2 + 2cd + d^2 - k^2 = (c + d)^2 - k^2$.
24. $4x^2 + 4xy + y^2 - z^2$.
25. $9a^2 + 12ab + 4b^2 - 16c^2$.
26. $25k^2 - 40kl + 16l^2 - 100m^2$.
27. $x^2 + y^2 - 2xy - 9z^2$.
28. $a^2 - (b - c)^2$.
29. $4x^2 - (3y + 2z)^2$.
30. $9a^2 - b^2 - c^2 + 2bc$. *Suggestion.* $9a^2 - b^2 - c^2 + 2bc = 9a^2 - (b^2 - 2bc + c^2) = 9a^2 - (b - c)^2$.
31. $l^2 - 9m^2 - 16n^2 - 24mn$.
32. $25a^2 - 25b^2 + 40bc - 16c^2$.
33. $36a^2x^2 - 9b^2y^2 - 49c^2z^2 - 42bcyz$.
34. $(x + y)^2 - (a + b)^2$. *Suggestion.* $(x + y)^2 - (a + b)^2 = [(x+y) + (a+b)][(x+y) - (a+b)]$.
35. $(4a + 3b)^2 - (5c - d)^2$.
36. $(a - 9)^2 - (a + 4)^2$.
37. $4x^2 + 9y^2 + 12xy - a^2 - 16 - 8a$.
38. $36a^2b - 24axy + 27a^3 + 12ab^2 - 3ay^2 - 48ax^2$.
39. $(5x - 7y)^2 - (2x + 3y)^2$.

40. $a^2 - b^2 - c^2 + 2\,bc.$

41. $4\,z^2 - x^2 + 2\,xy - y^2.$

42. $1 - a^2 + 2\,x - b^2 + x^2 - 2\,ab.$

43. $(x + 2\,y)^2 - (x - 3\,y)^2.$

44. $25 - (a - 3\,b)^2.$

45. $49\,a^2 - a^2x^2 - a^2y^2 - 2\,a^2xy.$

101. The Factors of the Sum or the Difference of Two Cubes. — The student should verify by division that

$$\frac{a^3 + b^3}{a + b} = a^2 - ab + b^2$$

and

$$\frac{a^3 - b^3}{a - b} = a^2 + ab + b^2,$$

and he should write out in ordinary language rules, based on the formulæ, for forming these quotients.

EXERCISES

Factor the following:

1. $x^3 - y^3.$
2. $x^3 + y^3.$
3. $a^3 - 1.$
4. $b^3 + 8.$
5. $m^3 - 27.$
6. $a^4 - ab^3.$
7. $x^6 + 2\,x^3y^3 + y^6.$
8. $64\,a^3 - 125\,b^3.$
9. $a^6 + b^6.$
10. $16\,m^3 - 54\,n^3.$
11. $27\,x^3 + 64\,a^3.$
12. $125\,a^3b^3c^3 + 64\,x^3y^3z^3.$
13. $a^6 - b^6.$

Suggestion. — This polynomial can be factored either as the difference of two cubes or as the difference of two squares. $a^6 - b^6 = (a^2 - b^2)(a^4 + a^2b^2 + b^4)$, or $a^6 - b^6 = (a^3 + b^3)(a^3 - b^3) = (a + b)(a^2 - ab + b^2)(a - b)(a^2 + ab + b^2)$. When a polynomial can be factored by either one of these rules, the latter is to be preferred.

14. $x^{12} - y^{12}.$
15. $5^3\,a^3 - 2^3\,b^3.$
16. $a^{12} + b^{12}.$
17. $(a + b)^3 + (c + d)^3.$
18. $x^9 - k^9.$
19. $64\,a^6 - 27\,x^{12}.$
20. $l^3m + m^4.$
21. $a^{12} - b^{12}.$
22. $27\,a^3 + x^3.$
23. $250\,a^3 + 16\,b^3.$
24. $5 + 40\,m^3n^3.$
25. $a^6 - 2\,a^3b^3 + b^6.$
26. $x^6 + 16\,x^3 + 64.$
27. $x^5 + x^3 + x^2 + 1.$

28. $b^{3m} - 1$.

29. $a^{3x} + 8$.

30. $x^{3m} + y^{3n}$.

31. $(x + 2)^3 + 1$.

32. $(a + b)^3 - 8$.

33. $(a - b)^3 + (x - y)^3$.

102. The Product of Two Binomials having a Common Term. — The student should verify by multiplication the formula:

$$(x + a)(x + b) = x^2 + (a + b)x + ab.$$

This, translated into ordinary language, says:

The product of two binomials having a common term is equal to the square of the common term, plus the sum of the unlike terms multiplied by the common term, plus the product of the unlike terms.

EXERCISES

Expand the following by this formula:

1. $(x + 1)(x + 2)$.
2. $(a - 3)(a + 2)$.
3. $(m - 5)(m - 7)$.
4. $(l + 4)(l - 3)$.
5. $(x - 3)(x - 10)$.
6. $(a^2 + 5)(a^2 - 3)$.
7. $(x + 5)(x + 4)$.
8. $(x - 9)(x - 10)$.
9. $(l - 6)(l + 5)$.
10. $(a + 20)(a - 3)$.
11. $(a + 20)(a + 3)$.
12. $(b - 2)(b - 12)$.
13. $(2x - 3)(2x - 10)$.
14. $(2x + 5)(2x + 6)$.
15. $(3x - 1)(3x - 2)$.
16. $(4xy - 2)(4xy + 3)$.
17. $(3x + 4)(3x + 4)$.
18. $(2k + 3)(2k + 5)$.
19. $(abc + 2)(abc - 5)$.
20. $(4x - 3y)(4x - 2y)$.
21. $(7a - 3b)(7a + 4b)$.
22. $(3r + 2s)(3r - 4s)$.
23. $(2x - 5)(2x - 6)$.
24. $(a + 2b)(a + 3b)$.
25. $(a + 2b)(a - 3b)$.
26. $(x + 4y)(x + y)$.
27. $(x - 4y)(x - 4y)$.
28. $(abc + 2)(abc - 5)$.
29. $(x^m + y)(x^m + 2y)$.
30. $(a^x - 5)(a^x - 6)$.
31. $[(b+c)-2][(b+c)+8]$.
32. $[(a+x)-1][(a+x)+2]$.
33. $[(m + 2n) + x][(m + 2n) + 3x]$.

Determine the following quotients by inspection:

34. $\dfrac{a^2+5a+6}{a+2}$.

35. $\dfrac{a^2-5a+6}{a-2}$.

36. $\dfrac{a^2+a-6}{a+3}$.

37. $\dfrac{a^2-a-6}{a-3}$.

38. $\dfrac{k^2+7k+12}{k+3}$.

39. $\dfrac{r^2+10r+16}{r+8}$.

40. $\dfrac{x^2+12x+35}{x+5}$.

41. $\dfrac{x^2-12x+35}{x-5}$.

42. $\dfrac{x^2+2x-35}{x+7}$.

43. $\dfrac{x^2-2x-35}{x-7}$.

44. $\dfrac{b^4+6b^2+8}{b^2+2}$.

45. $\dfrac{x^6+3x^3y^3+2y^6}{x^3+2y^3}$.

46. $\dfrac{a^8+5a^4+4}{a^4+4}$.

47. $\dfrac{4b^2+14bc+12c^2}{2b+3c}$.

48. $\dfrac{9x^2+18xy+5y^2}{3x+y}$.

103. The Factors of Trinomials of the Form x^2+bx+c.—We have already learned how to recognize and to factor trinomial squares. We now consider trinomials of the form

$$x^2+bx+c.$$

These can be factored if we can find two numbers k and l such that $kl=c$ and $k+l=b$. For in this case

$$(x+k)(x+l)=x^2+(k+l)x+kl=x^2+bx+c.$$

Suppose we wish to factor x^2+5x+6. We look for two numbers whose sum is 5 and whose product is 6. Obviously 2 and 3 are such numbers. Hence

$$x^2+5x+6=(x+2)(x+3).$$

Trinomial squares and polynomials that are the difference of two squares belong under this head, although they have been treated separately.

EXERCISES

Factor:

1. $x^2 - 5x + 6$.
2. $x^2 + x - 6$.
3. $x^2 - x - 6$.
4. $a^2 + 7a + 12$.
5. $a^2 - 7a + 12$.
6. $a^2 - a - 12$.
7. $a^2 + a - 12$.
8. $l^2 + 5l + 4$.
9. $m^2 + 11m + 18$.
10. $r^2 + 9r + 20$.
11. $r^2 + r - 20$.
12. $r^2 - r - 20$.
13. $a^4 + 4a^2 + 3$.
14. $x^2 + 18x + 45$.
15. $b^6 + 20b^3 + 64$.
16. $c^2 - 12c - 64$.
17. $y^2 - 2y - 63$.
18. $x^2 + 2x - 63$.
19. $a^2 - 16a - 105$.
20. $x^2 + 18x + 65$.
21. $a^2 + 12a + 35$.
22. $a^2 - 2a - 35$.
23. $a^2 + 2a - 35$.
24. $x^2 - x - 72$.
25. $x^2 + x - 72$.
26. $b^6 + 8b^3 + 15$.
27. $y^2 + 11y + 10$.
28. $z^2 - 9z - 10$.
29. $x^2 - 9$.
30. $x^{2a} - 5x^a + 6$.

104. The Product of the Binomials $ax + b$ and $cx + d$. — By performing the multiplication, we find that

$$(ax + b)(cx + d) = acx^2 + (ad + bc)x + bd.$$

The terms acx^2 and bd are called *end products*, and the terms adx and bcx are called *cross products*. It is easy to see that *the product of $ax + b$ and $cx + d$ is a polynomial whose terms are the end products and the sum of the cross products.*

Thus, $(2x + 3)(5x + 4) = 10x^2 + (8 + 15)x + 12 = 10x^2 + 23x + 12$.

EXERCISES

Make use of this rule to write down the following products:

1. $(x + 5)(3x + 4)$.
2. $(x - 1)(2x + 3)$.
3. $(5a + 3b)(a - 7b)$.
4. $(5a - 4)(a + 1)$.
5. $(2x - 1)(x - 1)$.
6. $(7r - 2)(4r - 3)$.
7. $(5a + 2)(2a - 5)$.
8. $(y - 8)(11y + 5)$.
9. $(2m - 7)(3m + 2)$.
10. $(3x - 4)(3x + 4)$.

11. $(9a-4)(2a-5)$.
12. $(3x+1)(7x+4)$.
13. $(2a+2)(a-5)$.
14. $(10y+1)(11y-1)$.
15. $(4z+3)(3z+4)$.
16. $(a+2)(2a-5)$.
17. $(3a-5b)(a+b)$.
18. $(5y+1)(y+3)$.
19. $(5a+2)(4a-3)$.
20. $(8x+3)(5x+8)$.
21. $(2x-5)(3x+5)$.
22. $(2k+7)(4k-3)$.
23. $(7a-4b)(3a+5b)$.
24. $(4x-5)(x+2)$.
25. $(x-9)(2x-3)$.
26. $(3r-4s)(2r+7s)$.
27. $(8x-5)(3x+8)$.
28. $(3x+y)(5x-y)$.
29. $(3x-y)(5x+y)$.
30. $(7abc+2)(5abc-3)$.
31. $(2m+3)(7m-2)$.
32. $(3x-1)(7x+4)$.
33. $(5m-7)(3m+2)$.
34. $(11y+5)(11y-5)$.
35. $(6x-4y)(2x+8y)$.
36. $(5a+2)(3a-5)$.
37. $(6m-5n)(6m-5n)$.
38. $(3r-7)(5r+2)$.
39. $(2x+5y)(3x+4y)$.
40. $(d+6)(6d+1)$.
41. $(2x+3)(x-1)$.
42. $(a-5)(2a+2)$.
43. $(3y+5)(y+1)$.
44. $(y+5)(3y+1)$.
45. $(3x+5)(2x+7)$.
46. $(3x+5)(2x-7)$.
47. $(3x-5)(2x+7)$.
48. $(3x-5)(2x-7)$.
49. $(2x+5)(3x+7)$.
50. $(5x+2)(7x-3)$.
51. $(5x-2)(7x+3)$.
52. $[3(x+y)+2][5(x+y)-1]$.
53. $[(a+b)-6][4(a+b)+1]$.
54. $[(c+d)+2][5(c+d)-3]$.

105. Factors of Trinomials of the Form ax^2+bx+c. — All the products in the exercises of the preceding article are of the form ax^2+bx+c.

If such a trinomial has two binomial factors, it is clear from the preceding article that one of the end products of the factors is ax^2, the other end product c, and the sum of the cross products bx; and we can take advantage of this fact to find the binomial factors of certain trinomials of this form. Consider the following examples:

Example 1. $6x^2+11x+3$.

The product of the first terms of the factors sought must be $6x^2$, and each of these first terms must contain x; the product of the second terms must be 3. The possible pairs of factors are therefore

$$\begin{array}{cccc} 6x+1 & 6x+3 & 3x+1 & 3x+3 \\ x+3 & x+1 & 2x+3 & 2x+1 \end{array}$$

The sum of the cross products of these respective pairs of factors is $19x$, $9x$, $11x$, and $9x$. The third one of these equals the middle term of the given polynomial. We conclude therefore that

$$6x^2+11x+3=(3x+1)(2x+3),$$

and it is easy to verify that this conclusion is correct.

Example 2. $10x^2-11x-6$.

If this trinomial has two binomial factors, the second terms of these factors must have opposite signs, since their product is -6. The possible pairs of factors are

$$\begin{array}{cccccc} 10x-6 & 10x+6 & 10x-1 & 10x+1 & 10x-3 & 10x+3 \\ x+1 & x-1 & x+6 & x-6 & x+2 & x-2 \\ 10x-2 & 10x+2 & & & & \\ x+3 & x-3 & & & & \end{array}$$

$$\begin{array}{cccccccc} 5x-6 & 5x+6 & 5x-1 & 5x+1 & 5x-3 & 5x+3 & 5x-2 & 5x+2 \\ 2x+1 & 2x-1 & 2x+6 & 2x-6 & 2x+2 & 2x-2 & 2x+3 & 2x-3 \end{array}$$

An examination of the cross products of these respective pairs of factors shows that in the last pair the sum of these cross products is $-11x$. We conclude therefore that

$$10x^2-11x-6=(5x+2)(2x-3).$$

After some practice the student should be able to detect the factors of polynomials of this kind without writing down in this way all the possible pairs of factors.

Not every trinomial of the form ax^2+bx+c can be factored in this way. For example, $2x^2+x+1$ has no factors of the kind we are here considering. We shall see later, however, that it is the product of two factors of a different kind. (See Second Course.)

EXERCISES

Factor the following trinomials:

1. $6x^2 - x - 2$.
2. $6x^2 + 11x + 3$.
3. $6a^2 + 7a + 2$.
4. $4a^2 + 13a - 35$.
5. $15m^2 + 8m - 12$.
6. $16k^2 + 22k - 3$.
7. $12l^2 - 7l - 12$.
8. $4y^2 + 13y - 12$.
9. $4y^2 + 10y + 6$.
10. $9a^2 + 15a + 4$.
11. $4b^2 + 16b + 15$.
12. $4a^2 - 4a - 15$.
13. $x^2 + x + 1$.
14. $4x^2 - 2x - 12$.
15. $4y^2 - 7y - 5$.
16. $9x^2 - 12x - 5$.
17. $3x^2 + 2x - 1$.
18. $16b^2 - 8b - 15$.
19. $9y^2 - 27y + 30$.
20. $16x^2y^2 - 20xy - 6$.
21. $5y^2 + 3y - 2$.
22. $9a^2 + 9a - 10$.
23. $16x^2 + 8xy - 15y^2$.
24. $25l^2 - 65lm + 42m^2$.
25. $2x^2 - 3xy - 9y^2$.
26. $6a^2 + ay - 15y^2$.
27. $9x^2 + 27xy + 20y^2$.
28. $10x^2y^2 + 29xyz + 10z^2$.
29. $10x^2y^2 - 29xyz + 10z^2$.
30. $10a^2b^2 + 21abc - 10c^2$.
31. $10a^2b^2 - 21abc - 10c^2$.
32. $-8x^2 - 26xy + 15y^2$.
33. $-3a^2 - 8ab - 4b^2$.
34. $3x^2 - 16x - 35$.
35. $3a^2 + 11a - 4$.
36. $4 - 7a + 3a^2$.
37. $5 + 26x + 5x^2$.
38. $10r^2 + 11r - 6$.
39. $6x^2 - 11x - 10$.
40. $6x^2 + 7xy + 4y^2$.
41. $6x^2 - 10xy - 4y^2$.
42. $4m^2 - 7mn - 15n^2$.
43. $3a^2 + 10ab + 3b^2$.
44. $7x^2 + 2a^2 - 9ax$.
45. $2a^2 + 9ax + 7x^2$.
46. $3x^2 + 4xy + y^2$.
47. $3(a+b)^2 + 5(a+b) - 2$.
48. $10(y+z)^2 - 17(y+z) + 3$.
49. $2(x-y)^2 - 11(x-y) + 5$.

106. Many expressions that we wish to factor do not come under any of the foregoing heads. Sometimes, as has been pointed out, the expression under consideration has no factors of the kind we are here considering. And sometimes, when there are such factors we must resort to ingenious devices in order to find them.

For example, if we wish to find the factors of $x^4 + x^2y^2 + y^4$, we see that it does not come under any of the forms we have discussed. But we can bring it under the form of the difference of two squares by a slight change in the way of writing it. Thus,

$$x^4 + x^2y^2 + y^4 = x^4 + 2x^2y^2 + y^4 - x^2y^2$$
$$= (x^2 + y^2)^2 - x^2y^2 = (x^2 + y^2 + xy)(x^2 + y^2 - xy).$$

If we wish to factor $a^3 + a + 10$, we observe that $a^3 + a + 10 = a^3 - 4a + 5a + 10 = a(a^2 - 4) + 5(a + 2) = (a + 2)(a^2 - 2a + 5)$.

EXERCISES

Factor:

1. $a^4 + a^2b^2 + b^4$.
2. $x^4 + x^2 + 1$.
3. $16x^4 + 4x^2y^2 + y^4$.
4. $81k^4 + 36k^2l^2 + 16l^4$.
5. $a^4 + 2a^2b^2 + 9b^4$.
6. $x^4 + x^2y^2 + 25y^4$.
7. $a^4 - 9a^2x^2 + 16x^4$.
8. $b^4 - 3b^2 + 9$.
9. $x^4 - 7x^2 + 1$.
10. $a^3 - a + 6$.
11. $b^3 - 21b + 20$.
12. $y^3 - 3y + 2$.

107. It is comparatively easy to factor an expression when we know to what typical form it belongs. But in practice these expressions are presented to us with nothing to indicate how they can be factored except the expressions themselves. Facility in recognizing these typical forms can be acquired by practice.

The following general directions may be helpful to the student:

If there is a monomial factor, express the polynomial as the product of a monomial and a polynomial.

If this latter polynomial is a binomial, see if it is the difference of two squares, or the sum of two cubes, or the difference of two cubes.

If it is a trinomial, apply the rules given for factoring trinomials.

If it has more than three terms, try to factor it by grouping its terms, or by any artifice that may occur to you. Proceed in the same way with each factor obtained until you feel assured that all the factors are prime. The product of all the factors found should equal the original polynomial.

MISCELLANEOUS EXERCISES

Factor the following polynomials:

1. $4a^2 - 2a - 20$.
2. $x^5 + 8x^2$.
3. $b^6 - 26b^3 - 27$.
4. $13a^4 - 118a^2b^2 + 240b^4$.
5. $3x(b - c) + 3x(m + n)$.
6. $a^2(3x - 5y) - a^2(4y - x)$.
7. $(r - s)^2 - 15(r - s) - 16$.
8. $4x^2(a + b) + 12xy(a + b) + 9y^2(a + b)$.
9. $6a^2 - 37ab + 6b^2$.
10. $(x^2 - 5x)^2 - 5(x^2 - 5x) + 4$.
11. $(a^2 + 7a)^2 + 3(a^2 + 7a) + 2$.
12. $(y^2 + 4y)^2 + 7(y^2 + 4y) + 12$.
13. $r^3 + r^2 - 9r - 9$.
14. $ab^3 + 1 + b + a$.
15. $b^4 + 2b^3 + 6b - 9$.
16. $a^2 + (x + 2y)a + 2xy$.
17. $2y^2 + 4by + 6cy + 12bc$.
18. $(x^2 + y^2)^2 - 4x^2y^2$.
19. $(x + y)^3 + (x - y)^3$.
20. $x^4 - x^3y - xy^3 + y^4$.
21. $1 - b^2c^2 - x^2y^2 + 2bcxy$.
22. $x^9 - x^6 - 64x^3 + 64$.
23. $a^3 - 12a + 16$.
24. $x^{2n} - 1$.
25. $x^{3n} + 1$.
26. $y^6 - 8y^3$.
27. $a(x - y) + 2a(m + n)$.
28. $a^2 + 2ab + b^2 - 9$.
29. $a^3 + 3a^2b + 3ab^2 + b^3 - 8$.
30. $x^6 - 9x^3 + 8$.
31. $2(a + b)^2 - (a + b) - 3$.
32. $a^6x^6 - 64$.
33. $36x^3 - 51x^2 - 15x$.
34. $(a + 1)^2 - (b - 1)^2$.
35. $6m^2 - 13mn + 6n^2$.
36. $a^4 + a^3 + a + 1$.
37. $x^3 - x^2 - 4x + 4$.
38. $x^{2a} - 7x^a + 10$.
39. $x^2 - 4y^2 - 9z^2 + 12yz$.
40. $27x^3 + 27x^2 + 9x + 1$.

41. $(2x-3y)^2+2(2x-3y)(3x-2y)+(3x-2y)^2$.

42. $a^4+a^2x^2+x^4$.

43. $b^{2n}-9$.

44. x^3-6x+9.

45. $(a+x)^3-(a-x)^3$.

46. $x^{3a}-3x^{2a}+3x^a-1$.

47. $a^2-13a+36$.

48. $2x^2+11x+15$.

49. $a^4+a^3b-ab^3-b^4$.

50. x^4+4x^2+16.

51. $x^{2n}+2x^ny^n+y^{2n}$.

52. $1-2y^n+y^{3n}$.

108. Solution of Equations in one Unknown by Factoring. — The student has learned how to identify a number from a description in the form of an equation of the first degree. But many of the descriptions he will have to consider are in the form of equations of the second or higher degrees, and it is important for him to know how to identify a number from such a description.

Suppose we have the description

$$3x^2-2x-8=0.$$

This is the same as $(x-2)(3x+4)=0$.

The left member of this equation (or description) is the product of two factors, and the right member is zero.

Now any number which, when put in place of x in this equation, makes either factor of the left member zero, answers the description given by the equation, and, conversely, any number that answers this description must, when put in the place of x, make one of these factors zero, since the product of the two factors cannot be zero unless at least one of the factors is zero.

Hence the numbers described by this equation are the same as the numbers described by the two equations

$$x-2=0$$

and

$$3x+4=0.$$

These numbers are 2 and $-\frac{4}{3}$.

From this example we derive the following general method of procedure for the solution of an equation of any degree in one unknown:

Rule. — *Transpose all the terms of the equation to the left member, making the right member zero; factor the resulting left member; put equal to zero each of these factors that contains the unknown, and solve the resulting equations.*

In practice this method is subject to limitation, because there are so many polynomials that we cannot factor. A method of procedure for the solution of equations of the second degree to which this method of factoring cannot be readily applied will be given in Chapter XV.

EXERCISES

Use the method of factoring given above to find the numbers described by each of the following equations. Check your results.

1. $x^2 - 3x + 2 = 0$.
2. $x^2 + 5x = 0$.
3. $x^2 + 10a^2 = 7ax$.
4. $2x^2 + 5x = 2x + 5$.
5. $10x^2 + 17mx + 3m^2 = 0$.
6. $(x + 1)(x^2 - 9) = 0$.
7. $x^3 + x^2 - 12x = 0$.
8. $x^4 - 29x^2 + 100 = 0$.
9. $3x^2 + 2x - 8 = 0$.
10. $y^3 - 3y^2 + 3y - 1 = 0$.
11. $z^2 - 2z + 1 = 9$.
12. $x^2 + 6x + 8 = 0$.
13. $(r + 3)^2 - 4 = 0$.
14. $(y + 1)(y^2 + 12y + 20) = 0$.
15. $(x^2 - x - 42)(x^2 - 8x + 15) = 0$.
16. $x^2 - x = 20$.
17. $4x^3 = 2x^2 + 20x$.
18. $y^3 + 8y^2 = 0$.
19. $x^3 - 12x + 16 = 0$.
20. $x^3 + x^2 = 9x + 9$.

21 $z^2 - 9z + 8 = 0$.

22. $4y^2 - 2y - 12 = 0$.
23. $4x^2 - 7x + 3 = 0$.
24. $9r^2 - 12r - 5 = 0$.
25. $6x^2 + 7x + 2 = 0$.
26. $12x^2 = 12 + 7x$.
27. $6m^2 + 11m + 3 = 0$.
28. $3x^2 + 11x - 4 = 0$.
29. $7x^2 = 9x - 2$.
30. $6x^2 + x - 15 = 0$.
31. $x^3 - 3x + 2 = 0$.
32. $x^3 - 21x + 20 = 0$.

109. Illustrative Problem.—An open box containing 144 cubic inches is to be made from a square piece of tin by cutting out 4-inch squares from the corners and turning up the sides. How large a piece of tin should be used?

Let x = number of inches in each side of the tin.

Then the base of the box will be a square each side of which is $(x-8)$ inches.

Hence there will be $4(x-8)^2$ cubic inches in the box and

$$4(x-8)^2 = 144.$$

Then $$(x-8)^2 - 36 = 0,$$

and $$(x-8+6)(x-8-6) = 0.$$

Hence $$x = 2 \text{ or } 14.$$

We conclude from this that a piece of tin 14 inches square should be used, since it would be impossible to cut out 4-inch squares from the corners of a piece of tin 2 inches square.

110. Usually in solving a problem the equation we get is a restatement of the conditions of the problem. But sometimes it is impossible to put all the conditions of the problem into the equation. In such cases it may be that some of the roots of the equation do not satisfy those conditions that were not put into it, and these roots must be rejected.

In the preceding problem, for example, it is implied that the piece of tin to be used must be more than 8 inches square. But it was impossible to put this condition into the equation. At the end, therefore, it was necessary to reject the 2 because it was less than 8.

PROBLEMS

1. Find two consecutive integers the sum of whose squares is 365.

2. Find two consecutive integers whose product is 240.

3. Find two consecutive even integers whose product is 168.

4. Find two consecutive odd integers whose product is 255.

5. Find two consecutive odd integers the sum of whose squares is 130.

6. Find all the numbers that are 210 less than their respective squares.

7. Find all the numbers that are two thirds of their respective squares.

8. An open box 3 inches deep and twice as long as it is wide is to be made from a piece of tin by cutting out equal squares from the corners and turning up the sides and ends. What should be the dimensions of the piece of tin used in order that the box contain 168 cubic inches?

9. An open box containing 108 cubic inches is to be made from a piece of tin 12 inches square by cutting out equal squares from the corners and turning up the sides and ends. How large should be the squares cut out from the corners?

Hint. — If $x =$ number of inches in the side of each of the squares to be cut out, the conditions of the problem are expressed by the equation

$$4x^3 - 48x^2 + 144x - 108 = 0.$$

The left member can be factored by grouping the terms thus:

$$4(x^3 - 27) - 48x(x - 3) = 4(x - 3)(x^2 - 9x + 9).$$

It is unnecessary for the solution of this problem to find the factors of $x^2 - 9x + 9$.

10. An open box containing 32 cubic inches is to be made from a rectangular piece of tin 6 by 12 inches by cutting out equal squares from the corners and turning up the sides and ends. How much should be cut out from each corner?

11. A man wishes to make a tin box in the way described in the preceding problem that shall be twice as wide as it is deep and three times as long as it is deep. How large a piece of tin should he use in order that the box contain 162 cubic inches?

Hint. — Let $x =$ the depth of the box in inches.

12. What should be the size of the square corners cut out from a piece of tin a inches square in order to form an open box whose sides should have the same area as the parts cut out?

13. What should be the dimensions of a rectangular card in order that it should contain 15 square inches and that the distance around it should be 16 inches?

14. What should be the dimensions of a rectangular lot containing an acre in order that the distance around it be 52 rods?

15. How wide should a path around the field of the preceding problem be in order that it should contain 27 square rods?

16. The owner of a rectangular lot 12 rods long and 5 rods wide wishes to enlarge it into a rectangular lot twice as large by increasing the length and the width the same amount. What should this increase be?

17. What must be the radius of a circle in order that a circle with a radius 3 inches shorter shall be four ninths as large?

18. What should be the sizes of two circles in order that the difference of their diameters be one inch and the difference of their areas be seven ninths the area of the smaller one?

19. A lever is to be cut from a bar weighing 3 pounds to the foot. How long must it be in order that it may balance about a point 4 feet from one end when a weight of 18 pounds is attached to this end?

20. A lever is to be cut from a bar weighing 2 pounds to the foot. How long must it be in order that it may balance about a point 3 feet from one end when a weight of 9 pounds is attached to this end?

CHAPTER X

HIGHEST COMMON FACTOR AND LOWEST COMMON MULTIPLE

111. Highest Common Factor. — In the applications of arithmetic the student has had frequent occasion to find the greatest number that is a divisor of each of two or more numbers. He will recall that such a number is called the *greatest common divisor* of the given numbers.

In order to find the greatest common divisor of any given number, we proceed in the following way:

1. *Factor each of the given numbers into its prime factors.*
2. *Form the product of those prime factors that occur in all the numbers, taking each one the least number of times it occurs in any of the numbers.*

This product is the greatest common divisor of the given numbers.

Example. Find the greatest common divisor of 54, 162, and 180.

$$54 = 2 \cdot 3^3,$$
$$162 = 2 \cdot 3^4,$$
$$180 = 2^2 \cdot 3^2 \cdot 5.$$

The greatest common divisor is therefore $2 \cdot 3^2 = 18$.

EXERCISES

Find the G. C. D. (greatest common divisor) of the following numbers:

1. 24, 30.
2. 30, 80.
3. 25, 35, 42.
4. 91, 75.
5. 12, 15, 18.
6. 25, 40.
7. 125, 15.
8. 27, 108, 135.

9. 30, 40, 90.
10. 144, 24, 45.
11. 18, 27, 108, 522.
12. 4224, 525.
13. $2^3 \cdot 7^2 \cdot 11^4$, $3^5 \cdot 7^3 \cdot 11^3$.
14. 1728, 81, 5280.
15. 336, 1134.
16. 684, 304.
17. $2^3 \cdot 3^4 \cdot 5^2$, $3^5 \cdot 5^3 \cdot 7^3$.
18. 775, 620, 312.
19. $5^4 \cdot 11^3$, $5^3 \cdot 11^4 \cdot 13$.
20. $3^2 \cdot 11^4 \cdot 17$, $3 \cdot 11^3 \cdot 17^2$.

112. Two or more polynomials may have more than one common factor of the same degree. For example, $2x^2 - 2y^2$ and $2x^2 + 4xy + 2y^2$ have the common factors $x + y$ and $2(x + y)$, which are of the first degree. Of all the common factors of two or more polynomials the one which is of the highest degree and which is divisible by the greatest common divisor of the numerical coefficients of the polynomials is of the greatest importance. It is called the **highest common factor** of the polynomials.

We use the term *highest* here instead of *greatest* in order to emphasize the fact that we are interested in the factor of highest degree, irrespective of the numerical value.

The common factor of highest degree may or may not be the common factor with the greatest numerical value.

Thus, in the example of § 113 $x + y$ is a common factor of the given polynomials and $x(x + y)$ is the highest common factor. But when $x = \frac{1}{2}$ and $y = 3$, for example, $x + y$ has a greater value than $x(x + y)$.

113. The following rule for finding the highest common factor of two or more polynomials is obvious:

Rule. — 1. *Factor each of the given polynomials into its prime factors.*

2. *Form the product of those prime factors that occur in all the polynomials, taking each one the least number of times it occurs in any of the polynomials.*

This product is the highest common divisor of the given polynomials.

Note. — The student should compare this rule with the rule for finding the greatest common divisor of two or more numbers.

Definition. — Two polynomials that have no common factor except 1 are said to be *relatively prime*.

Example. — Find the highest common factor of $x^3 - xy^2$, $x^4 + xy^3$, and $x^3 + 3x^2y + 2xy^2$.

$$x^3 - xy^2 = x(x+y)(x-y),$$
$$x^4 + xy^3 = x(x+y)(x^2 - xy + y^2),$$
$$x^3 + 3x^2y + 2xy^2 = x(x+y)(x+2y).$$

The highest common factor is therefore $x(x+y)$.

But $-x(x+y)$ is also a common factor of the highest degree and is divisible by the greatest common divisor of the numerical coefficients of the polynomials. It is, therefore, also a highest common factor of these polynomials. As a matter of fact, any polynomials have two highest common factors that differ only in sign.

EXERCISES

Find the H. C. F. (highest common factor) of the following polynomials:

1. $24a^3$, $16a^4$.
2. $4x^3y^2$, $6x^2y^4$.
3. $12l^4m^3n^2$, $9l^3m^4n^4$.
4. $36a^2b^3x$, $27l^4c^3$.
5. $10A^3B^2$, $25AB^3$, $40A^2B^4$.
6. $64x^3y^3z^3$, $96x^2y^4z^3$, $256x^2y^2z^5$.
7. $6rst$, $15r^3s^2$, $21s^3t^2$.
8. $12hkl$, $18h^2k^2l^2$, $25h^3k^3l^3$.
9. $a^2 - b^2$, $b^3 - a^3$.
10. $x^2 + 5x + 6$, $x^2 + x - 2$.
11. $(h+2k)^2(h-k)^3(3h+4k)$, $(h+2k)^3(h-k)^2(2h-3k)$.
12. $x^3 + 7x^2 + 6x$, $x^4 + x$.
13. $m^2 - 7m + 12$, $bm - 3b - am + 3a$.
14. $r^2 + 4r + 4$, $r^2 - 6r - 16$.
15. $27 + x^3$, $x^2 + 6x + 9$.
16. $x^3 - y^3$, $x^2 - y^2$, $x^2 - 2xy + y^2$.
17. $r^5 - rs^4$, $r^3 + r^2s + rs + s^2$.

18. $a^2 + 3ab + 2b^2,\ a^2 + 5ab + 6b^2,\ a^2 + ab - 2b^2.$

19. $x^2 + 7x - 30,\ x^2 + 11x - 42,\ x^2 - x - 6.$

20. $6a - 10ay + 4ay^2,\ 18a - 8ay^2,\ 54a - 16ay^3.$

21. $4(a + b) - x(a + b),\ a(a + b) - 3b(a + b).$

22. $3bx^2 + 3x^2y,\ ab + bx + ay + xy.$

23. $4a^2 + 4y^2 - 8ay,\ a^3 - 3a^2y + 3ay^2 - y^3.$

24. $x^2 + 3x + 2,\ x^2 + 10x + 9.$

25. $-x^2 - 2xy - y^2,\ x^2 + 2xy + y^2.$

26. $5x^2 - 5y^2,\ x^2 + 3xy + 2y^2,\ x^2 + 2xy + y^2.$

27. $a^6 - b^6,\ a^4 - b^4,\ a^2 - b^2.$

28. $a^6 + b^6,\ a^4 + b^4,\ a^2 + b^2.$

29. $a^3 - ab^2,\ a^2 - 6ab + 5b^2,\ (a - b)^4.$

30. $x^3 - 27,\ x^2 - 9,\ x^2 - 6x + 9.$

31. $x^6 + 2x^3y^3 + y^6,\ x^2 + 2xy + y^2.$

32. $(a + b)^3 + (c + d)^3,\ (a + b)^2 - (c + d)^2.$

33. $x^6 + y^6,\ x^6 - y^6.$

34. $a^{12} - b^{12},\ a^9 - b^9.$

35. $a^2 - 16a - 105,\ a^2 + 9a + 20,\ a^2 + 6a + 5.$

36. $x^2 - 7x + 12,\ x^3 - 3x^2 - 4x + 12.$

37. $a^4 + 9a^2 + 81,\ a^4 - 27a.$

38. $9x^2 - 12x - 5,\ 6x^2 + 11x + 3.$

39. $5y^2 + 3y - 2,\ 15y^2 + 14y - 8.$

40. $3x^2 + 2x - 1,\ x^2 + 2x + 1,\ x^2 - 1$

41. $a^{2n} + 3a^n + 2,\ a^{2n} + 4a^n + 3.$

42. $2x^{2b} + 7x^b + 6,\ x^{2b} + 5x^b + 6.$

43. $a^{3x} + 3a^{2x} + 3a^x + 1,\ a^{2x} + 2a^x + 1.$

114. Lowest Common Multiple. — The student will also recall that in his study of arithmetic in connection with the subject of the greatest common divisor he had occasion to find the least number that was divisible by each of two or more given numbers. Such a number was called the *least common multiple* of the given numbers. It is the product of all the prime factors of the given numbers, each prime factor being taken the greatest number of times it occurs in any of the numbers.

Example. — Find the L. C. M. (least common multiple) of 54, 162, 180.

$$54 = 2 \cdot 3^3,$$
$$162 = 2 \cdot 3^4,$$
$$180 = 2^2 \cdot 3^2 \cdot 5.$$

The L. C. M. is therefore $2^2 \cdot 3^4 \cdot 5 = 1620$.

EXERCISES

Find the L. C. M. of the numbers in Ex. 1–20, § 111.

In our application of algebra we shall need to know how to solve the analogous algebraic problem; namely, to find the polynomial of lowest degree that is divisible by each of two or more polynomials. This polynomial is called the **lowest common multiple** of the given polynomials, and is found by a process closely analogous to the process for finding the L. C. M. of two or more numbers.

Exercise. — Formulate the rule for finding the L. C. M. of two or more polynomials.

Example. — Find the L. C. M. of the polynomials $r^2 + 5r + 6$, $r^3 + 8$, and $r^4 - 8r^2 + 16$.

$$r^2 + 5r + 6 = (r + 2)(r + 3),$$
$$r^3 + 8 = (r + 2)(r^2 - 2r + 4),$$
$$r^4 - 8r^2 + 16 = (r + 2)^2(r - 2)^2.$$

Hence the L. C. M. is $(r + 2)^2(r - 2)^2(r + 3)(r^2 - 2r + 4)$.

EXERCISES

Find the L. C. M. of the following polynomials:

1. a^2b, ab^2, a^3b^3.
2. $6m^3n^2, 10mn^4, 12mn$.
3. $8xy^2z^3, 12x^3yz^2, 20x^2y^3z$.
4. $6, 7ab^3c, 3a^2b, 6a^3b^2$.
5. $9, 12r^3s^2, 15r^4$.
6. $3x^2, x^3 - xy^2$.
7. $12, 8A^2, 10A^3 + 30A^2 + 20A$.
8. $x + y, x - y, x^2 - y^2$.
9. $a^2 - b^2, a^2 + 2ab + b^2$.
10. $4(b + c)^2, 2(b + c)^3$.
11. $x^3(1 + x)^2(1 - x), (1 + x)(1 - x)$.
12. $2(x - y)^3, 3(x - y)(x + y)^3$.
13. $a^3 + b^3, a^3 - b^3, a^2 - b^2, a^2 + b^2$.
14. $2x^2 + 9x + 7, x^2 + 6x + 5$.
15. $2a^2 - a - 10, 2a^2 + a - 3$.
16. $3ab^2 - 21ab + 36a, b^2 - 5b + 4$.
17. $r^2 + 2rs + s^2, r^2 - 2rs + s^2, r^2 - s^2$.
18. $xy - 4x + yz - 4z, ax + xy + az + yz$.
19. $l^2 + lm + ln + mn, l^2 + 2lm + m^2$.
20. $x^4 - 2x^2y^2 + y^4, x^6 - y^6$.
21. $x^2 - y^2, x^2 + y^2$.
22. $a^4 + a^2b^2 + b^4, a^3 + b^3$.
23. $b^3 - x^3, x^3 - b^3$.
24. $49x^8 - 64y^8, 49x^8 - 112x^4y^4 + 64y^8$.
25. $a^3 - b^3, a + b$.
26. $x^3 + 8, x^2 - 2x + 4, 3x^2 + 7x + 2$.
27. $x^2 - 7x + 12, x^3 - 3x^2 - 4x + 12$.
28. $a^3 - a^2 + 9a - 9, a^3 - 3a^2 + 3a - 1$.
29. $a^6 + x^6, a^2 + x^2$.

30. x^3-4x^2+2x+1, $8x^2-4x-2x^3-2$.

31. y^3+7y^2+6y, y^4+y.

32. x^3-y^3, $x^2-2xy+y^2$.

33. $(a-3b)^2(a+5b)^4$, $3(a-3b)^3(a+5b)$.

34. x^3+6x^2+x+6, $x^3+7x^2+14x+8$.

35. x^2+4x+4, $x^3+6x^2+12x+8$.

36. $64-a^3$, $16-a^2$.

37. $3(x+y)-a(x+y)$, $x^2+3xy+y^2$.

38. $(a-b)(b-c)$, $(b-c)(c-a)$, $(a-c)(b-a)$.

39. $(x-1)(x-2)$, $(x-2)(x-3)$, $(x-3)(x-1)$.

40. $3x^2-16x-35$, $6x^2+7x-5$.

41. $2x^2-3xy-9y^2$, $4x^2-3xy-10y^2$.

42. $a^{2n}+3a^n+2$, $3a^{2n}+5a^n+2$.

43. $b^{2x}-c^{2y}$, $b^{2x}+2b^xc^y+c^{2y}$.

44. $a^{3x}+3a^{2x}+3a^x+1$, a^x+1.

CHAPTER XI

FRACTIONS

115. We shall be unable to make the fullest possible use of algebra as a means of discovering the information implied in certain statements unless we know something about handling the indicated quotients of polynomials.

Definition.— Such indicated quotients, and indicated quotients that can be put into this form without changing their values, are called **rational fractions.**

Since every polynomial represents a number it is to be expected that the rules for handling these fractions should be very similar to those for handling arithmetic fractions.

Definition. — The dividend of an indicated division is called the **numerator** of the fraction and the divisor is called the **denominator.**

The numerator is usually written over the denominator with a horizontal line between them, as in arithmetic. Thus $\frac{a^2+3ab+2b^2}{a^2+7ab+4b^2}$.

The student is familiar with the fact that $\frac{4}{6}=\frac{2}{3}$. Now this equality holds by virtue of a certain principle which also applies to algebraic fractions and which may be stated as follows:

Principle. — *The numerator and the denominator of a fraction can be multiplied by the same expression or divided by the same expression without changing the value of the fraction.*

Thus,
$$\frac{a+2b}{a+b}=\frac{(a+2b)(a+3b)}{(a+b)(a+3b)}=\frac{a^2+5ab+6b^2}{a^2+4ab+3b^2},$$

and
$$\frac{x^2-4}{x^3-8}=\frac{(x-2)(x+2)}{(x-2)(x^2+2x+4)}=\frac{x+2}{x^2+2x+4}.$$

116. Since in accordance with the Principle stated in § 115 the form of a fraction can be changed in many ways without changing its value, it is important to know the form most suitable for a given purpose. Now for many purposes the most suitable form is that in which the numerator and the denominator have no common factor. When a fraction is in this form it is said to be in its **lowest terms.**

For example, $\frac{x-y}{x+y}$ is obviously in its lowest terms, while $\frac{x^2-y^2}{x^2+2xy+y^2}$ is not in its lowest terms.

Principle. — *A fraction can be reduced to its lowest terms by dividing its numerator and its denominator by their highest common factor.*

But usually it is not necessary to find this highest common factor. The desired result can be obtained by applying the following

Rule. — *Find the prime factors of the numerator and of the denominator and divide each by the factors common to the two.*

Thus,
$$\frac{x^2-y^2}{x^2+2xy+y^2}=\frac{(x-y)(x+y)}{(x+y)(x+y)}=\frac{x-y}{x+y}.$$

117. Common terms of the numerator and the denominator must not be stricken out, since this is equivalent to subtracting the same thing from both numerator and denominator. Such an operation performed on the numerator and the denominator of a fraction in general changes its value.

Thus, $\frac{3+1}{2+1}=\frac{4}{3}$. But when we strike out the common term 1 from the numerator and the denominator the resulting fraction is $\frac{3}{2}$, which is not equal to $\frac{4}{3}$. Likewise the common term $4x$ cannot be stricken out from the numerator and the denominator of the fraction $\frac{7a+4x+3}{3b+4x+1}$ without changing the value of the fraction.

The student will be inclined to make mistakes of this kind on the false assumption that operations which can be performed on the two members of an equation without affecting the description given by the equation (§ 64) can also be performed on the numerator and the denominator of a fraction without changing the value of the fraction. Some operations that can be performed without changing the value of the fraction will be mentioned in § 120.

EXERCISES

Reduce the following fractions to their lowest terms:

1. $\frac{18}{24}$. 2. $\frac{9}{80}$. 3. $\frac{15}{25}$. 4. $\frac{21}{80}$. 5. $\frac{72}{100}$.

6. $\dfrac{5 \cdot 7^2 \cdot 13}{5^2 \cdot 7 \cdot 11}$.

7. $\dfrac{2^4 \cdot 3^2 \cdot 5^3}{3^3 \cdot 5 \cdot 7^2}$.

8. $\dfrac{5^3 \cdot 3^2}{9 \cdot 125}$.

9. $\dfrac{x^3y}{x^2y^4}$.

10. $\dfrac{10\,a^2b^6}{16\,ab^7}$.

11. $\dfrac{4\,x^2y}{6\,xy^2}$.

12. $\dfrac{8\,p^2q^3r}{20\,q^4}$.

13. $\dfrac{12\,l^3mn^2}{25\,x^3yz^2}$.

14. $\dfrac{4\,x}{6\,x+6}$.

15. $\dfrac{a^2+7\,a+12}{a^2+4\,a+3}$.

16. $\dfrac{2\,\pi r(r+h)}{2\,\pi r(r+H)}$.

17. $\dfrac{a^3-b^3}{2\,a^2-3\,ab+b^2}$.

18. $\dfrac{3\,x^2-12\,y^2}{3\,x^2+9\,xy+6\,y^2}$.

19. $\dfrac{r^2-25}{r^2-r+30}$.

20. $\dfrac{a^2-ab-3\,a+3\,b}{(a^2-b^2)(a-3)}$.

21. $\dfrac{a^2-b^2}{(a-b)^2}$.

22. $\dfrac{x^3+y^3}{(x+y)^3}$.

23. $\dfrac{l^2-m^2}{l^2+m^2}$.

24. $\dfrac{x^3-27}{x^2-9}$.

25. $\dfrac{a^6-b^6}{a^2-b^2}$.

26. $\dfrac{r^3+R^3}{R^2+8\,Rr-7\,r^2}$.

27. $\dfrac{a^4-b^4}{a^6-b^6}$.

28. $\dfrac{2^4 \cdot 3^2 \cdot 7^3(x^3-y^3)(x^3+y^3)}{2^5 \cdot 3^3 \cdot 7^4(x^6-y^6)}$.

29. $\dfrac{(x^2-7x+10)(x^2-5x+6)}{(x^2-25)(x^2-6x+9)}$.

30. $\dfrac{2^4 \cdot 5^3 - 2^2 \cdot 5^4}{2^3 \cdot 5^3 - 2 \cdot 5^4}$.

31. $\dfrac{3^2 \cdot 5^3 \cdot 7 - 3^3 \cdot 5^3 \cdot 7^2}{5^2 \cdot 7^3 \cdot 13 - 5^3 \cdot 7^2 \cdot 11}$.

32. $\dfrac{x^4 + x^2y^2 + y^4}{x^6 - y^6}$.

33. $\dfrac{a^2 + a - 6}{a^2 + 8a + 15}$.

34. $\dfrac{a^2 + 2ab + b^2 - c^2}{a^2 + 2ac + c^2 - b^2}$.

35. $\dfrac{b^3 - 3b^2 + 3b - 1}{b^2 - 1}$.

36. $\dfrac{y^6 - 8y^3}{y^2 - 5y + 6}$.

37. $\dfrac{(a+b)^2 - (x-y)^2}{(a+b+x)^2 - y^2}$.

38. $\dfrac{(m-n)^2 - 7(m-n) + 12}{(m-n)^2 - 6(m-n) + 9}$.

39. $\dfrac{x^4 + x^3 - x - 1}{x^6 - 1}$.

40. $\dfrac{b^{3n} - x^{3n}}{b^{2n} - x^{2n}}$.

41. $\dfrac{x^{3m} + 1}{x^{2m} - x^m + 1}$.

118. In the discussion of division in §§ 56 and 57 consideration of the case in which a factor occurs in the divisor to a higher power than in the dividend was deferred. From what he has already learned in this chapter, the student can see that the result of a division can be indicated by a fraction irrespective of the relative powers of the factors of the dividend and the divisor.

119. Addition and Subtraction of Fractions. — The sum of two or more fractions with the same denominator is a fraction with the same denominator, the numerator of which is the sum of the numerators of the given fractions.

Thus,

$$\frac{3}{7}+\frac{2}{7}=\frac{5}{7};\quad \frac{x-y}{x+y}+\frac{3}{x+y}+\frac{-2x+3y}{x+y}=\frac{x-y+3-2x+3y}{x+y}=\frac{-x+2y+3}{x+y}.$$

If we wish to add two or more fractions that do not have the same denominator, we must first reduce them to equivalent fractions with the same denominator. This can be done as follows:

1. *Find the lowest common multiple of the denominators of the fractions.*
2. *Divide this lowest common multiple by the denominator of each fraction.*
3. *Multiply the numerators and denominators of the fractions by the respective quotients.*
4. *Form new fractions whose numerators and denominators are the respective products.*

The resulting fractions are equivalent to the original ones and all have the same denominator.

In practice, after we have divided the lowest common multiple of the denominators by the denominator of one of the fractions, it is unnecessary to perform the multiplication of this denominator by the resulting quotient since we know that the product must be the L. C. M. of the denominators.

Instead of taking the lowest common multiple of the denominators, we could have taken any common multiple of them. But the former way is the simpler one.

Definition. — The L. C. M. of the denominators of two or more fractions is called the **lowest common denominator** of the fractions.

Example 1. — Reduce $\frac{3}{2x}+\frac{4x+2}{x^2}+\frac{5x-1}{3x^3}$ to a single fraction.

Here the lowest common denominator is $6x^3$

$$\frac{3}{2x}=\frac{9x^2}{6x^3}$$

$$\frac{4x+2}{x^2}=\frac{24x^2+12x}{6x^3}$$

$$\frac{5x-1}{3x^3}=\frac{10x-2}{6x^3}$$

$$\frac{3}{2x}+\frac{4x+2}{x^2}+\frac{5x-1}{3x^3}=\frac{33x^2+22x-2}{6x^3}.$$

Example 2. — Reduce $\frac{3}{a+b}+\frac{2b}{a-b}+\frac{3a+5b}{a^2-b^2}$ to a single fraction.

The lowest common denominator is a^2-b^2.

$$\frac{3}{a+b}=\frac{3a-3b}{a^2-b^2}$$

$$\frac{2b}{a-b}=\frac{2ab+2b^2}{a^2-b^2}$$

$$\frac{3a+5b}{a^2-b^2}=\frac{3a+5b}{a^2-b^2}$$

$$\frac{3}{a+b}+\frac{2b}{a-b}+\frac{3a+5b}{a^2-b^2}=\frac{2ab+2b^2+6a+2b}{a^2-b^2}.$$

Algebraic reductions of this kind can be checked by substituting numbers for the letters. But these numbers must be so selected that they make no denominator zero. In the last example for $a=2$, $b=1$, we get

$$\frac{3}{a+b}=1 \qquad \frac{2ab+2b^2+6a+2b}{a^2-b^2}=\frac{4+2+12+2}{3}=\frac{20}{3}.$$

$$\frac{2b}{a-b}=2 \qquad \text{And} \qquad 1+2+\frac{11}{3}=\frac{20}{3}.$$

$$\frac{3a+5b}{a^2-b^2}=\frac{11}{3}$$

The fact that the two expressions we said were equal have equal numerical values for certain values of the letters makes it probable that our statement was correct.

EXERCISES

Reduce each of the following expressions to a single fraction, and check some of your results:

1. $\frac{2}{3}+\frac{3}{5}$. 3. $\frac{3}{4}-\frac{5}{6}$. 5. $1\frac{2}{3}-5\frac{1}{2}+7\frac{1}{3}$.

2. $\frac{3}{7}+\frac{1}{21}+\frac{1}{3}$. 4. $\frac{3}{4}+\frac{2}{7}-\frac{1}{3}$. 6. $\frac{5}{3}-\frac{11}{2}+\frac{57}{8}$.

7. $\frac{3a}{2}+\frac{5b}{3}$. 10. $\frac{2b+5}{3}+\frac{7-4b}{6}$.

8. $\frac{2x}{3}-\frac{5x}{4}+\frac{x}{5}$. 11. $\frac{3}{r^2}+\frac{4}{r}+\frac{8}{r^3}$.

9. $\frac{4m+3}{5}+\frac{1-2m}{7}$. 12. $\frac{5}{3^2\cdot 4}+\frac{7}{3\cdot 4^2}+\frac{35}{3^3\cdot 4^2}$.

13. $\frac{x}{2y}+\frac{y}{3x}+\frac{3y^2}{x^2}$.

14. $\frac{7}{4a^2b^3}+\frac{3}{5ab}+\frac{2}{3a^3b^2}+\frac{1}{12ab^4}$.

15. $\frac{8}{xy}+\frac{5b}{x^2}+\frac{3a}{5y^2}$.

16. $\frac{3a-2}{a^2}+\frac{7-4a^2}{5a^3}+\frac{a^3-4}{10a^4}$.

17. $\frac{4x^2-9}{5x^2}+\frac{x^2-4}{xy^2}+\frac{3-x}{2xy}$.

18. $\frac{3}{x^2-16}+\frac{1}{2}+\frac{5}{x^2-6x+8}$.

19. $\frac{a-b}{(a+b)^2}+\frac{a-b}{a^2-b^2}$.

20. $\frac{l-4}{l^2-6l}+\frac{2l-3}{l^2-8l+12}$.

21. $\frac{a^2-3ab+b^2}{9-6a+a^2}+\frac{5a-3y}{9-3a}$.

22. $\frac{r+s}{r-s}+\frac{r-s}{r+s}$.

23. $\frac{r-5}{r-6}+\frac{3r}{r+2}+\frac{1}{r^2-4r-12}$.

24. $\frac{x+y}{2}+\frac{1}{x+y}+\frac{1}{x-y}$.

25. $\frac{2a}{(a+b)^2}+\frac{3b}{(a-b)^2}$.

26. $\frac{a}{b}+\frac{b}{a}+\frac{2}{a+b}$.

27. $\frac{2a}{a+5}+\frac{1}{a+6}$.

28. $\frac{4}{x-2y}+\frac{5}{x-3y}$.

29. $\frac{2a-3b}{a+b}+\frac{1}{4a}+\frac{2}{3b}$.

30. $\frac{x^2+2x}{x+1}+\frac{2}{x-1}+\frac{3x+2}{x^2-1}$.

31. $\frac{3x^2+2}{x^2+4x}+\frac{3x}{x^2-16}+\frac{x+3}{x-4}$.

32. $\frac{m^2+n^2}{m^3-n^3}+\frac{m^2-n^2}{m^3-n^3}+\frac{2}{m^2-mn+n^2}$.

33. $\frac{1}{x^2+xy+y^2}+\frac{1}{x^2-xy+y^2}$.

34. $\frac{3x^2+x-1}{x^3+x^2+2x+2}+\frac{3}{x+1}+\frac{4}{x^2+2}$.

35. $\frac{x+4}{x^3-2x^2+2x-1}+\frac{3}{x^2-x+1}$.

36. $\frac{x-y}{x^2+3xy+2y^2}+\frac{3y-2x}{x^2+2xy+y^2}$.

37. $\frac{a^2-b^2}{a^2+b^2}+\frac{a^4+b^4}{a^4-a^2b^2+b^4}$.

38. $\frac{2m-3n}{m^2-n^2}+\frac{3}{m-n}$.

39. $\frac{4c+5d}{c^2+cd+d^2}+\frac{3c}{c-d}$.

40. $\frac{x^3+27y^3}{x^2+3xy+9y^2}+\frac{3x-8y}{x-3y}$.

41. $\frac{y+5}{y^2-2y}+\frac{4}{y+2}+\frac{6y}{y^2-4}$.

42. $\frac{a+2b}{a-2b}+3a-5b+\frac{a-2b}{a+2b}$.

120. Changes in a Fraction that do not alter its Value. — 1. *The value of a fraction is not changed if the sign of every term of the numerator and every term of the denominator is changed.*

This is because the changing of these signs has the same effect as multiplying both numerator and denominator by -1.

Thus, $$\frac{-2}{3}=\frac{2}{-3};\quad \frac{a^2-ab+b^2}{-a+b}=\frac{-a^2+ab-b^2}{a-b}.$$

2. *The value of a fraction merely changes its sign when the sign of every term of the numerator is changed; and also when the sign of every term of the denominator is changed.*

This is an immediate consequence of the rule of signs for division.

Thus, $$\frac{-2}{3}=-\frac{2}{3};\quad \frac{a^2-ab+b^2}{-a+b}=-\frac{-a^2+ab-b^2}{-a+b}=-\frac{a^2-ab+b^2}{a-b}.$$

The student is familiar with the fact that the difference of two numbers is equal to the sum of the minuend and the subtrahend with its sign changed. It is clear then that the difference of two fractions is equal to the sum of the minuend and the subtrahend with the sign of every term of its numerator changed.

Thus, $$\frac{2a^2}{a^2-b^2}-\frac{a+b}{a^2-b^2}=\frac{2a^2-a-b}{a^2-b^2}.$$

In practice the student will find it advisable to inclose the numerator of the subtrahend in parentheses with a minus sign before it, instead of changing the sign of every term of this numerator.

Example: —

$$\frac{x-1}{x+1}+\frac{x^2-5}{x^2-1}-\frac{x+1}{x-1}=\frac{(x-1)(x-1)+(x^2-5)-(x+1)(x+1)}{x^2-1}$$

$$=\frac{x^2-2x+1+x^2-5-x^2-2x-1}{x^2-1}=\frac{x^2-4x-5}{x^2-1}$$

$$=\frac{(x-5)(x+1)}{(x+1)(x-1)}=\frac{x-5}{x-1}.$$

This example calls attention to the fact that in adding two or more fractions we sometimes get a result that is not in its lowest terms. In such a case we should always reduce the result to its lowest terms.

After the student acquires a certain degree of familiarity with this kind of work he will find the arrangement of the work adopted in this example preferable to that used in § 119.

The student should be on his guard against confusing the sign of the first term of the numerator or the first term of the denominator with the sign before the fraction. Moreover, if the numerator has more than one term, he must not suppose that the effect of changing the sign of the first term is counterbalanced by changing the sign before the fraction, and in changing the sign of every term of the numerator he must not overlook the first term. Similar precautions apply also to the denominator.

3. *The value of a fraction is not changed if the signs of an even number of factors of the numerator, or of the denominator, or of the two together, are changed.*

This is an immediate consequence of the law of signs for multiplication.

Thus, $$\frac{(x-y)(x+2y)(x-3y)}{(x+y)(x-2y)(x+5y)}=\frac{(y-x)(x+2y)(3y-x)}{(x+y)(x-2y)(x+5y)}.$$

Do not confuse factors and terms.

4. *The value of a fraction merely changes its sign when the signs of an uneven number of factors of the numerator, or of the denominator, or of the two together, are changed.*

This is also an immediate consequence of the law of signs for multiplication.

$$\frac{(x-y)(x+2y)(x-3y)}{(x+y)(x-2y)(x+5y)}=-\frac{(x-y)(x+2y)(x-3y)}{(x+y)(2y-x)(x+5y)}.$$

121. Definition. — A **mixed expression** is the indicated sum of a polynomial and one or more fractions.

Thus, $3a + 2b + \frac{2a - b}{a + b}$ and $x + 2y - \frac{4xy}{x + 3y} + \frac{2x - y}{x - y}$ are mixed expressions.

Mixed expressions can be reduced to single fractions in accordance with the preceding rules if we look upon the polynomials as fractions whose denominators are 1.

EXERCISES

Reduce the following expressions to single fractions and check some of your results:

1. $\frac{1}{x} + \frac{1}{y}$.
2. $\frac{1}{x} - \frac{1}{y}$.
3. $\frac{1}{x} + \frac{1}{y} - \frac{2}{xy}$.
4. $\frac{3r-1}{6r^2} - \frac{7-2r^2}{4r^3} + \frac{5r^3-8}{9r^4}$.
5. $\frac{1}{r_1} - \frac{1}{r_2}$.
6. $\frac{1}{R} + \frac{1}{R'}$.
7. $\frac{1}{f} + \frac{1}{f'} - \frac{a}{ff'}$.
8. $\frac{x}{y} + \frac{y}{x} - 1$.
9. $2a - 3b + \frac{1}{4a} - \frac{2}{3b}$.
10. $\frac{1}{r_1^2} + \frac{2}{r_1 r_2} + \frac{1}{r_2^2}$.
11. $\frac{a}{a - b} + \frac{b}{b - a}$.
12. $\frac{2a^2}{a^2 - b^2} - \frac{b}{a - b} - \frac{b}{a + b}$.
13. $\frac{m - n}{(m + n)^2} - \frac{m - n}{m^2 - n^2}$.
14. $\frac{3}{y + 5} - \frac{2}{y - 5}$.
15. $\frac{a + b}{a - b} - \frac{a - b}{a + b} + 2$.
16. $\frac{x^2 - y^2}{x + y} - x + y$.
17. $\frac{r + s}{r - s} + \frac{r - s}{r + s} - 2$.
18. $a - 1 - \frac{a + 1}{a - 1}$.

19. $\dfrac{b-1}{2-b}-\dfrac{b^2+3}{b^2-4}.$

20. $\dfrac{3x}{x^2-4}-\dfrac{1}{x+2}+\dfrac{3}{2-x}.$

21. $\dfrac{1}{a+b}+3(a+b).$

22. $x^2-y^2-\dfrac{3x^3+y^3}{x-y}+4xy.$

23. $\dfrac{2}{1-x^3}+\dfrac{5x}{1-x}+\dfrac{3x+2}{1+x+x^2}.$

24. $\dfrac{x-3}{x^2-3x+2}-\dfrac{x-1}{x^2-5x+6}+\dfrac{2x}{x^2-4x+3}.$

25. $\dfrac{5}{x^2}-\dfrac{2}{y^2}-\dfrac{2x}{x-y}+\dfrac{-4x}{x+y}.$

26. $\dfrac{1}{(a-1)(a+2)}+\dfrac{1}{(a+2)(a+3)}+\dfrac{1}{(a+3)(a-1)}.$

27. $\dfrac{1}{(x-y)(y-z)}-\dfrac{1}{(z-y)(y-x)}+\dfrac{1}{(z-x)(y-x)}.$ Be sure to get the lowest common denominator.

28. $\dfrac{A^3}{1-A^3}-\dfrac{2A}{A^2+A+1}+\dfrac{1}{A-1}.$

29. $1+\dfrac{r_4}{r_2}+\dfrac{r_4}{r_1}-\dfrac{r_4r_3}{r_3+r_1{}^2}.$

30. $\dfrac{1}{(a-b)(b-c)}+\dfrac{1}{(b-c)(c-a)}+\dfrac{1}{(c-a)(a-b)}.$

31. $\dfrac{1}{f-d}-\dfrac{1}{f'}.$

32. $\dfrac{a-b}{a^3+b^3}+\dfrac{2a+5b}{(a+b)^3}.$

33. $\dfrac{2x}{x^2-3}+\dfrac{5x}{3-x^2}+\dfrac{4x-1}{x^2+2}.$

34. $\dfrac{5}{3x+2y}-\dfrac{3}{3x-2y}-\dfrac{x+y}{4y^2-9x^2}.$

35. $\dfrac{3}{(x-2)(x-3)}+\dfrac{2}{(x-3)(2-x)}.$

36. $\frac{x^2+2x+4}{x^2-2x+4}-\frac{x-2}{x+7}-\frac{4x-1}{x^3+8}$.

37. $x-\frac{3}{x^2+2xy+y^2}+2y$.

38. $-\left(x-2-\frac{5x}{x+2}\right)-\left(5x+6-\frac{3}{x+3}\right)$. Remove parentheses first.

39. $\left(3l+2m-4+\frac{l+m}{l+2m}\right)-\left(\frac{l-m}{l+3m}-4l+5m+2\right)$.

40. $-\left(3x+4y-\frac{3}{x+y}\right)+\left(\frac{-3}{x+y}+4y+3x\right)$.

41. $2b+3c+\frac{4b}{2b+c}-\left(4b+3c-\frac{4b}{3b-2c}\right)$.

122. Reduction of a Fraction to a Mixed Expression. — We have seen how to reduce a mixed expression to a single fraction, and this latter form is for many purposes the more convenient one. Sometimes, however, the former is the more convenient and it becomes desirable to change a fraction to a mixed expression. This may occur when the degree of the numerator in some letter is equal to, or greater than, the degree of the denominator in the same letter.

Example. — Reduce $\frac{x^3+1}{x-1}$ to a mixed expression.

Divide the numerator by the denominator.

$$\begin{array}{l|l}
x^3+1 & x-1 \\ \cline{2-2}
\underline{x^3-x^2} & x^2+x+1 \\
\quad x^2+1 & \\
\quad \underline{x^2-x} & \\
\qquad x+1 & \\
\qquad \underline{x-1} & \\
\qquad\quad 2 &
\end{array}$$

Here the dividend is x^3+1, the divisor $x-1$, the quotient x^2+x+1, and the remainder 2. Hence

$$x^3+1=(x^2+x+1)(x-1)+2, \quad \text{(See § 89)}$$

and therefore
$$\frac{x^3+1}{x-1}=x^2+x+1+\frac{2}{x-1}.$$

Rule.—1. *Arrange the numerator and the denominator according to the descending powers of any letter whose degree in the numerator is at least equal to its degree in the denominator.*

2. *Divide the numerator by the denominator and continue the division until the remainder is of lower degree in this letter than is the denominator.*

3. *The fraction is then equal to the quotient plus the fraction whose numerator is the remainder and whose denominator is the denominator of the original fraction.*

EXERCISES

Reduce the following fractions to mixed expressions:

1. $\frac{5}{2}$.
2. $\frac{9}{4}$.
3. $\frac{28}{11}$.
4. $\dfrac{-21}{10}$.
5. $\frac{156}{20}$.
6. $\dfrac{x^3-1}{x^3+3x}$.
7. $\dfrac{x^2-3x+3}{(x-1)(x+2)}$.
8. $\dfrac{5x^3+1}{x^2-3x+2}$.
9. $\dfrac{a+b}{a-b}$.
10. $\dfrac{a-b}{a+b}$.
11. $\dfrac{3x+2}{x+5}$.
12. $\dfrac{y^2+6y+4}{y^2-2y+1}$.
13. $\dfrac{x^3+1}{x-1}$.
14. $\dfrac{a^4+b^4}{a^2-b^2}$.
15. $\dfrac{b^2+b+1}{b^2-b+1}$.
16. $\dfrac{x}{x-1}$.
17. $\dfrac{x^3+7x^2+6x-6}{x^2+2x}$.
18. $\dfrac{x^2+2x+2}{(x-1)(x-2)}$.
19. $\dfrac{x^3}{x^2+7x+12}$.
20. $\dfrac{x^2-1}{x^2-4}$.
21. $\dfrac{a^6+b^6}{a^2-b^2}$.
22. $\dfrac{x^2}{x-1}$.
23. $\dfrac{(a-b)^3}{a^3-b^3}$.
24. $\dfrac{x^4+x^2+1}{x^2+x+1}$.
25. $\dfrac{16m^4-n^4}{2m-n}$.

26. $\dfrac{x^3+6x^2+7x-7}{x^2-3x}$.

27. $\dfrac{2x^3+8x^2-4x-12}{x^2-4x-5}$.

28. $\dfrac{a^2}{(a+1)^2}$.

29. $\dfrac{x^3-1}{(x+1)(x^2+2)}$.

30. $\dfrac{a^6-b^6}{a^2+ab+b^2}$.

31. $\dfrac{x^6-y^6}{x^4+x^2y^2+y^4}$.

32. $\dfrac{x^3+5x}{x^3-8}$.

33. $\dfrac{x+2y}{x-2y}$.

34. $\dfrac{a^2+b^2}{2a}$.

35. $\dfrac{a^2+ab+b^2}{a+b}$.

36. $\dfrac{x^2}{x^2+6x+5}$.

37. $\dfrac{x^2+2xy+3y^2}{x^3+y^2}$.

38. $\dfrac{x^2}{x^2-9}$.

39. $\dfrac{x^2-9}{x^2}$.

40. $\dfrac{a^2-2ab+b^2}{(a-b)^2}$.

123. Multiplication of Fractions. — The product of two or more fractions is a fraction whose numerator is the product of the numerators of the fractions and whose denominator is the product of their denominators.

If we want the product in its lowest terms, it is desirable to divide out all factors common to any numerator and any denominator before performing the multiplication.

Thus if we wish to find the product of $\dfrac{x^3+y^3}{x^3-y^3}$ and $\dfrac{x^2+xy-2y^2}{x^2+6xy+5y^2}$, we proceed as follows:

$$\frac{x^3+y^3}{x^3-y^3}\cdot\frac{x^2+xy-2y^2}{x^2+6xy+5y^2}=\frac{\cancel{(x+y)}(x^2-xy+y^2)}{\cancel{(x-y)}(x^2+xy+y^2)}\cdot\frac{(x+2y)\cancel{(x-y)}}{(x+5y)\cancel{(x+y)}}$$

$$=\frac{(x+2y)(x^2-xy+y^2)}{(x+5y)(x^2+xy+y^2)}=\frac{x^3+x^2y-xy^2+2y^3}{x^3+6x^2y+6xy^2+5y^3}.$$

Check. — Let $x=2$, $y=1$.

$$\frac{x^3+y^3}{x^3-y^3}=\frac{9}{7};\qquad \frac{x^2+xy-2y^3}{x^2+6xy+5y^3}=\frac{4}{21}.$$

Hence the value of the product ought to be $\frac{9}{7}\cdot\frac{4}{21}=\frac{12}{49}$. Now

$$\frac{x^3+x^2y-xy^2+2y^3}{x^3+6x^2y+6xy^2+5y^3}=\frac{12}{49}.$$

This indicates that the work is correct. Why not put $x=2$, $y=2$?

124. The general rule of procedure for forming the product of fractions or mixed expressions is as follows:

Rule. — 1. *If any factor is a mixed expression, reduce it to a single fraction.*

2. *Factor every numerator and every denominator into its prime factors and divide out all factors that are common to any numerator and any denominator.*

3. *The product sought is a fraction whose numerator is the product of the remaining factors in the numerators and whose denominator is the product of the remaining factors in the denominators.*

Step 2 of this rule may be omitted, but when this is done the product obtained may not be in its lowest terms.

The student is now in a position to appreciate the grounds upon which rests the Principle of § 115. We can illustrate what we mean by considering the examples given under this principle.

$$\frac{a+2b}{a+b}=\frac{a+2b}{a+b}\cdot 1=\frac{a+2b}{a+b}\cdot\frac{a+3b}{a+3b}=\frac{a^2+5ab+6b^2}{a^2+4ab+3b^2};$$

$$\frac{x^2-4}{x^3-8}=\frac{(x-2)(x+2)}{(x-2)(x^2+2x+4)}=\frac{x-2}{x-2}\cdot\frac{x+2}{x^2+2x+4}$$

$$=1\cdot\frac{x+2}{x^2+2x+4}=\frac{x+2}{x^2+2x+4}.$$

That is, the principle referred to rests upon the fact that a fraction whose numerator and denominator are the same is equal to 1 and upon the rule for the multiplication of fractions.

EXERCISES

Perform the following indicated multiplications:

1. $\frac{5}{6}\cdot\frac{2}{3}$.

2. $\frac{4}{9}\cdot\frac{27}{10}\cdot\frac{162}{48}$.

3. $\frac{65}{72}\cdot\frac{75}{81}\cdot\frac{162}{13}$.

4. $\dfrac{3^3\cdot 5^2}{2^4\cdot 7^3}\cdot\dfrac{7^2\cdot 8}{5\cdot 6^2}\cdot\dfrac{10\cdot 4^2\cdot 7^2}{3^2\cdot 5}$.

5. $\dfrac{3a^4x^3}{4y^2}\cdot\dfrac{12y^3}{15ax}$.

6. $\dfrac{5a}{9b}\cdot\dfrac{8ca}{12m}\cdot\dfrac{16m^2b}{21a^2}$.

7. $\left(-\dfrac{5x^3y}{2ab}\right)^2(-3)^4\left(\dfrac{a^2b^2}{2xy^2}\right)^3$.

8. $\frac{abc}{xyz} \cdot \frac{x^2}{a^2} \cdot -\frac{y^3}{b^3} \cdot -\frac{z^3}{c^3} \cdot \frac{ab^2c^2}{xy^2z^2}.$

9. $\frac{6\,ax+4\,bx}{3\,y^2} \cdot \frac{9\,y^3}{15\,a+10\,b}.$

10. $\frac{3\,A}{A^3-B^3} \cdot \frac{A^2+3\,AB-4\,B^2}{6\,AB}.$

11. $\frac{y^2-y}{y^3-1} \cdot \frac{3\,y^2+3\,y+3}{y^2+y-2}.$

12. $\frac{a^2+3\,ab+2\,b^2}{a^2+2\,ab-3\,b^2} \cdot \frac{a^2+ab-6\,b^2}{a^2-2\,ab-3\,b^2} \cdot \frac{a^2-4\,ab+3\,b^2}{a^2-4\,b^2}.$

13. $\frac{f-a}{f} \cdot \frac{f'+f-a}{f'(f-a)}.$

14. $(n-1)\left(\frac{1}{r}+\frac{1}{r'}\right).$

15. $2\,e\left(\frac{e}{2\,r}+1\right).$

16. $\frac{(a+b)^2}{(a+2\,b)^2} \cdot \frac{a^2+6\,ab+8\,b^2}{a^2+3\,ab+2\,b^2}.$

17. $\frac{x^2-9}{x^3+27} \cdot \frac{2}{x^2+9} \cdot \frac{x+3}{x-3}.$

18. $\frac{x^2+6\,x-7}{x^2-4} \cdot \frac{x^2-9}{x^2-6\,x+7}.$

19. $\frac{a}{a^2+b^2} \cdot \frac{a^2+b^2}{a}.$

20. $\left(r+\frac{rs+s^2}{r-3\,s}\right) \cdot \frac{r^2-6\,rs+9\,s^2}{2}.$

21. $\frac{x^3-1}{x^2-x+1} \cdot \frac{x^3+1}{x-1}.$

22. $\frac{x^2+5\,x+6}{x^2+7\,x-8} \cdot \frac{x^2+11\,x+18}{x^2+11\,x+30}.$

23. $\frac{a^3-27}{9-a^2} \cdot (4-5a^2) \cdot \frac{a^2+6\,a+9}{5\,a^2+4}.$

24. $\left(1-\frac{x^2}{y^2}\right)\frac{1}{x-y}.$

25. $\left(r+\frac{1}{r}\right)\left(r-\frac{1}{r}\right).$

26. $\left(1+\frac{4}{x}+\frac{3}{x^2}\right) \cdot \frac{3\,x}{x^2+2\,x+1}.$

27. $\left(a-\frac{b^2}{a}\right)\left(\frac{a}{b}+\frac{b}{a}\right).$

28. $\left(x+\frac{xy}{x+y}\right)\left(x-\frac{xy}{x+y}\right) \cdot \frac{x^3-y^3}{x^3+y^3}.$

29. $\left(\frac{a+b}{a-b}\right)^3 \cdot \frac{a^3-b^3}{a^3+b^3}.$

30. $(x^2+5\,x+6) \cdot \frac{x+4}{x^2+8\,x+15}.$

31. $\left(\frac{b}{c}+1\right)^2\left(\frac{b}{c}-1\right)^2.$

32. $\frac{1}{x^3-y^3}\cdot\frac{1}{x^3+y^3}.$

33. $\frac{x^3+2x^2y}{xy^2-y^3}\cdot\frac{y^3-xy^2}{(x+2y)^3}.$

34. $\frac{a-b}{(b-c)(c-a)}\cdot\frac{b-c}{(c-a)(a-b)}\cdot\frac{c-a}{(a-b)(b-c)}.$

35. $\frac{1}{(x-y)(y-z)}\cdot\frac{1}{(y-z)(x-z)}\cdot\frac{1}{(z-x)(y-x)}.$

36. $\left(x+\frac{1}{x}\right)\left(x^2+\frac{1}{x^2}\right).$

37. $abc\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right).$

38. $(2x-3y)\left(\frac{1}{2x}-\frac{1}{3y}\right).$

39. $\frac{a^3-b^3}{a^3+b^3}\cdot\frac{a+b}{a-b}\cdot\frac{(a-b)^2+ab}{(a+b)^2-ab}.$

40. $\frac{x^4-y^4}{x^2-4y^2}\cdot\frac{x^6-64y^6}{x^2+y^2}.$

125. Division of Fractions. — The student may have noticed that the rule just given for the multiplication of fractions is the same as the rule for the multiplication of fractions in arithmetic; and the rule for division is also the same as in arithmetic. We can accordingly state the

Principle. — *If we divide one fraction by another, the quotient is a fraction whose numerator is the product of the numerator of the first fraction and the denominator of the second, and whose denominator is the product of the denominator of the first fraction and the numerator of the second.*

With this principle in mind the following rule is obvious:

Rule. — *In order to divide one fraction by another one, invert the divisor and proceed as in multiplication.*

If either the dividend or the divisor is a mixed expression, it must be reduced to a single fraction before this rule is applied.

EXERCISES

Perform the following indicated divisions:

1. $\frac{2}{3} \div \frac{1}{4}$.

2. $\frac{9}{10} \div \frac{1}{10}$.

3. $\frac{81}{100} \div \frac{2}{5}$.

4. $\frac{3^3 \cdot 5^2}{2^4 \cdot 7^3} \div \frac{3^2 \cdot 5}{10 \cdot 4^2 \cdot 7^2}$.

5. $\frac{x}{3} \div \frac{x}{3}$.

6. $\frac{3\,a^2b}{7\,c^2} \div \frac{2\,ab^2}{5\,c}$.

7. $\left(\frac{5\,x^3y^2}{2\,ab}\right)^3 \div -10$.

8. $\frac{3\,\pi r}{8\,d^2} \div \frac{3\,r^2}{2\,d^2}$.

9. $\frac{x^2+3\,xy+2\,y^2}{10\,xy} \div \frac{x^2+3\,xy+2\,y^2}{xy}$.

10. $\frac{3\,R}{R^3-8} \div \frac{(R-2)(R^2+2\,R+4)}{R}$.

11. $\frac{x^3-1}{x^2-x+1} \div \frac{x^3+1}{x-1}$.

12. $\frac{a^2+3\,ab+b^2}{a^2+2\,ab-3\,b^2} \div \frac{a^2-4\,b^2}{a^2-4\,ab+3\,b^2}$.

13. $\frac{a^2+ax+ay+xy}{a^2-ax-ay+xy} \div \frac{a^2-y^2}{a^2-x^2}$.

14. $\frac{(x+4)(x-4)}{(x-3)(x-2)} \cdot \frac{(x+2)(x-5)}{(x-5)(x-6)} \div \frac{(x+4)(x-13)}{(x-6)(x+11)}$.

15. $\frac{m+n}{mn} \cdot \frac{m^2-n^2}{5(m^3+n^3)} \div \frac{(m-n)(m+n)}{5\,m^3x+5\,mx^3}$.

16. $\dfrac{x^2+3x+2}{x^2+10x+25} \div (x+2)^2.$

17. $\left(1-\dfrac{k^3}{l^3}\right) \div \dfrac{k-l}{3kl}.$

18. $\left(\dfrac{x^2}{y^2}-\dfrac{y^2}{x^2}\right) \div \left(\dfrac{x}{y}-\dfrac{y}{x}\right).$

19. $\left(4-4\dfrac{b}{a}-3\dfrac{b^2}{a^2}\right) \div \left(a-\dfrac{9b^2}{4a}\right).$

20. $\left(1-\dfrac{10}{x}+\dfrac{21}{x^2}\right) \div \dfrac{x^2-10x+21}{x^2}.$

21. $\left(\dfrac{n'}{n}W-W'\right) \div \left(\dfrac{n'}{n}-\dfrac{E'^2}{E^2}\right).$

22. $\left(W-\dfrac{n}{n'}W'\right) \div \left(1-\dfrac{n'}{n}\right).$

23. $\left(\dfrac{1}{x}-\dfrac{1}{y}\right) \div \left(\dfrac{1}{x}+\dfrac{1}{y}\right).$

24. $\dfrac{F}{a+b} \div \dfrac{F-c}{v}.$

25. $\left(a+\dfrac{1}{a}\right) \div \left(\dfrac{1}{a}-a\right).$

26. $\dfrac{x^2-7x+10}{x+3} \div \dfrac{x^2-25}{3x}.$

27. $\left(x^2-xy+y^2-\dfrac{x^3-y^3}{x+y}\right) \div \dfrac{1}{x+y}.$

28. $\left(\dfrac{a}{b}+\dfrac{b}{a}\right) \div \left(\dfrac{a^2}{b^2}+\dfrac{b^2}{a^2}\right).$

29. $\dfrac{3a^2}{a+1} \div \dfrac{2b^3}{b+1}.$

30. $\dfrac{a^2+3a+2}{a^2+6a+8} \div \dfrac{(a+1)^3}{a^3+8}.$

31. $(x^4+x^3-x-1) \div \dfrac{x^3-1}{x+1}.$

32. $\left(\dfrac{1}{x}+\dfrac{1}{y}+\dfrac{1}{z}\right) \div \dfrac{x+y+z}{xyz}.$

33. $\left(\dfrac{x+y}{x-y}+\dfrac{x-y}{x+y}+1\right) \div \dfrac{(x+y)^2}{(x-y)^2}.$

34. $\left(\dfrac{a}{b}+\dfrac{b}{a}-1\right) \div \left(\dfrac{a}{b}+\dfrac{b}{a}+1\right).$

35. $\dfrac{(x+1)^3}{(x-1)^3} \div \dfrac{x^3+1}{x^3-1}.$

36. $\left(\dfrac{a^3}{b^3}-1\right) \div \left(\dfrac{a}{b}-1\right).$

37. $\left[\dfrac{(x+y)^2}{(x-y)^2}-\dfrac{2(x+y)}{x-y}+1\right] \div \left(\dfrac{x+y}{x-y}-1\right).$

38. $\left[\left(\dfrac{m+2n}{m-n}\right)^2-\left(\dfrac{m-2n}{m+n}\right)^2\right] \div \left(\dfrac{m+2n}{m-n}-\dfrac{m-2n}{m+n}\right).$

39. $\left(\dfrac{1}{x^2}+\dfrac{2}{xy}+\dfrac{1}{y^2}\right) \div \left(\dfrac{1}{x^2}-\dfrac{2}{xy}+\dfrac{1}{y^2}\right).$

40. $\left(a-b+\dfrac{b^2}{a+b}\right) \div \left(a+b-\dfrac{a^2}{a-b}\right).$

126. Complex Fractions. — Not all the rational fractions of algebra are so simple as those we have been considering. Either the numerator or the denominator of a rational fraction or both of them may be mixed expressions or rational fractions instead of polynomials.

Thus, $\dfrac{3x+\dfrac{2}{x}}{x^2-3}$, $\dfrac{4a^2-3a+2b}{\dfrac{2}{a^2}+\dfrac{3}{ab}-\dfrac{1}{b^2}}$, and $\dfrac{\dfrac{1+x}{x}}{1-\dfrac{1}{x^2}}$.

Such fractions are called *complex fractions.*

Since a fraction is an indicated division, a complex fraction can be simplified in accordance with the following

Rule. — 1. *If either the numerator or the denominator of the fraction is not a polynomial, reduce it to an equivalent single fraction.*

2. *Simplify the resulting fraction in accordance with the rule for the division of fractions.*

Example. —

$$\frac{\dfrac{a}{1+a}+\dfrac{1-a}{a}}{\dfrac{a}{1+a}-\dfrac{1-a}{a}}=\frac{\dfrac{a^2+1-a^2}{a(1+a)}}{\dfrac{a^2-1+a^2}{a(1+a)}}=\frac{\dfrac{1}{a(1+a)}}{\dfrac{2a^2-1}{a(1+a)}}=\frac{1}{a(1+a)}\cdot\frac{a(1+a)}{2a^2-1}=\frac{1}{2a^2-1}.$$

Check. — Put $a = 2$.

Then

$$\frac{\dfrac{a}{1+a}+\dfrac{1-a}{a}}{\dfrac{a}{1+a}-\dfrac{1-a}{a}}=\frac{\dfrac{2}{3}+\dfrac{-1}{2}}{\dfrac{2}{3}-\dfrac{-1}{2}}=\frac{\dfrac{1}{6}}{\dfrac{7}{6}}=\frac{1}{7}.$$

$$\frac{1}{2a^2-1}=\frac{1}{8-1}=\frac{1}{7}.$$

Frequently a complex fraction can be simplified more directly by multiplying its fractional numerator and denominator by the lowest common multiple of their denominators. If we apply this process to the preceding illustrative example, we get

$$\frac{\dfrac{a}{1+a}+\dfrac{1-a}{a}}{\dfrac{a}{1+a}-\dfrac{1-a}{a}}=\frac{a^2+1-a^2}{a^2-1+a^2}=\frac{1}{2a^2-1}.$$

EXERCISES

Simplify the following fractions:

1. $\dfrac{1+\frac{2}{3}}{\frac{1}{5}}$.

2. $\dfrac{\frac{4}{5}}{\frac{3}{8}}$.

3. $\dfrac{5\frac{1}{2}}{3\frac{1}{4}}$.

4. $\dfrac{10\frac{1}{2}-3\frac{2}{3}}{7\frac{1}{4}+5\frac{1}{6}}$.

5. $\dfrac{\dfrac{x}{y}}{\dfrac{x^2}{y^2}}$.

6. $\dfrac{a^3}{\dfrac{1}{a^2}}$.

7. $\dfrac{\dfrac{W}{W+V}}{WV}$.

8. $\dfrac{1+\dfrac{x}{y}}{1-\dfrac{x}{y}}$.

9. $\dfrac{\dfrac{1}{m}-1}{R}$.

10. $(m-1)\left(\dfrac{1}{R_1}+\dfrac{1}{R_2}\right)$.

11. $$\frac{nE}{\frac{nr}{m}+R}.$$

12. $$\frac{\frac{1}{r-h}}{\frac{1}{r+h}}.$$

13. $$\frac{\frac{m-n}{m}-\frac{m+n}{n}}{\frac{m-n}{n}+\frac{m+n}{m}}.$$

14. $$\frac{\frac{a^2}{b^2}-\frac{b^2}{a^2}}{a+\frac{b^2}{a}}.$$

15. $$\frac{x-1+\frac{6}{x-6}}{x-2+\frac{3}{x-6}}.$$

16. $$\frac{V}{1+\frac{t}{273}}.$$

17. $$\frac{\frac{1}{x}+\frac{1}{y}}{\frac{1}{x}-\frac{1}{y}}.$$

18. $$\frac{75\left(1+\frac{30}{5550}\right)}{1+\frac{15}{5550}}.$$

19. $$\frac{1}{1+\frac{1}{x}}.$$

20. $$\frac{1}{1+\frac{1}{1+\frac{1}{x}}}.$$

21. $$\frac{VS}{4\pi\left(t-h+\frac{h}{k}\right)}.$$

22. $$\frac{E}{r+\frac{R}{m}}.$$

23. $$\frac{3E}{\frac{3R}{2}+r}.$$

24. $$\frac{ne}{\frac{n^2r}{N}+l}.$$

25. $$\frac{ne}{\frac{nr}{\frac{N}{n}}+l}.$$

26. $$\frac{E}{\frac{R}{2}+r+l}.$$

27. $$\frac{(1+x)^2}{1+\frac{1}{x}}.$$

28. $$\frac{a+\frac{b}{a}}{a-\frac{b}{a}}.$$

29. $$\frac{x+2+\frac{6}{x-3}}{x-4-\frac{8}{x-2}}.$$

30. $\dfrac{x+y}{\dfrac{1}{x}+\dfrac{1}{y}}.$

31. $\dfrac{a+\dfrac{1}{a}-1}{a+\dfrac{1}{a}+1}.$

32. $\dfrac{\dfrac{5}{x}+\dfrac{4}{3}}{x+2}.$

33. $\dfrac{a+\dfrac{a^2+b^2}{b}}{a^2-\dfrac{b^3}{a}}.$

34. $\dfrac{\dfrac{1}{x}}{1+\dfrac{1}{x}}.$

35. $\dfrac{\dfrac{x}{y}}{z}\div\dfrac{x}{\dfrac{y}{z}}.$

CHAPTER XII

EQUATIONS IN ONE UNKNOWN INVOLVING FRACTIONS

127. Now that we know something about algebraic fractions, it is desirable that we consider equations containing fractions with denominators involving the unknown. Equations of this kind present themselves in many important applications.

In general, the first step in the solution of such equations is to clear them of fractions. This is done by multiplying both members of the equation by the L. C. M. of the denominators. After we have cleared of fractions we can proceed as described in the rule in § 65.

The result can be checked as before.

Example 1. Solve the equation

$$\frac{5}{x}+\frac{4}{3}=\frac{9}{x}.$$

The L. C. M. of the denominators is $3x$. Multiplying both members of the equation by $3x$, we get

$$\begin{aligned} 15+4x&=27,\\ 4x&=27-15,\\ 4x&=12,\\ x&=3. \end{aligned}$$

Check.

$$\tfrac{5}{3}+\tfrac{4}{3}=\tfrac{9}{3}.$$

Example 2. Solve the equation $\dfrac{3}{x+1}=\dfrac{2}{x-1}$.

The L. C. M. of the denominators is x^2-1.

Multiplying both members of the equation by x^2-1, we get

$$\begin{aligned} 3(x-1)&=2(x+1),\\ 3x-2x&=2+3,\\ x&=5. \end{aligned}$$

Check.

$$\frac{3}{5+1}=\frac{1}{2},$$

$$\frac{2}{5-1}=\frac{1}{2}.$$

EXERCISES

Solve the following equations and check each solution:

1. $\frac{17}{x}=4.$

2. $\frac{3}{2y}=\frac{1}{5}.$

3. $\frac{x+5}{4}-\frac{2x+4}{2}+3=0.$

4. $\frac{1}{x}+\frac{2}{x}+\frac{3}{x}=5.$

5. $\frac{42}{r}-\frac{1}{r}+\frac{1}{3r}=20.$

6. $\frac{x+1}{x}+\frac{2}{3x}=\frac{8}{3}.$

7. $\frac{a}{3y}-\frac{a}{5y}=\frac{1}{15}.$ (Solve for y.)

8. $\frac{2a+x}{3x}=\frac{5a-x}{2x}+\frac{2}{9}.$ (Solve for x.)

9. $\frac{2a-3x}{6a}-\frac{5a+2x}{5a}+\frac{5}{6}=0.$ (Solve for x.)

10. $\frac{V+v}{V}=\frac{b}{b-h}.$ (Solve for V.)

11. $\frac{d-p}{pd}=f.$ (Solve for d.)

12. $\frac{1}{p}+\frac{1}{p'}=\frac{1}{f}.$ (Solve for p.)

13. $(n-1)\left(\frac{1}{R}+\frac{1}{R'}\right)=\frac{1}{p}+\frac{1}{p'}.$ (Solve for R.)

14. $D=\frac{760\,p(1+at)}{.31\,v(H-h)}.$ (Solve for v.)

15. $n'=n\left(1-\frac{s}{c}\right).$ (Solve for c.)

16. $n=\frac{Q}{606.5+.305\,t}.$ (Solve for t.)

128. In the cases we have considered so far, the equation resulting from clearing another equation of fractions has given the same description of the unknown as the original equation. But when we clear of fractions by multiplying both members by an expression involving the unknown the resulting description may differ from the original one in that there are more numbers answering it. Or, as we say, the new equation may have at least one root that is not a root of the original equation.

Consider, for example, the equation $\frac{4x-1}{2x-2}=\frac{6x-3}{2x-2}+2$.
Clearing of fractions, we get

$$\begin{aligned} 4x-1&=6x-3+4x-4,\\ 6x&=6,\\ x&=1. \end{aligned}$$

But when we substitute this value of x for x in the left member of the original equation we get $\frac{4-1}{2-2}=\frac{3}{0}\cdot$ Now, in accordance with our definition of division, $\frac{3}{0}$ does not represent any number.

A similar situation arises when we put 1 for x in the second member. Thus neither member of the equation has any value at all when 1 is put for x, and it would be obviously improper under these circumstances to say that the two members of the equation have the same value. As a matter of fact, the equation has no root.

If we clear the equation of fractions by multiplying both members by a common denominator other than the lowest common denominator, additional roots are usually introduced into the equation.

Consider, for example, the equation $\frac{2x}{x-1}+\frac{4x-1}{1-x}+1=0$.

Multiplying both members by $(x-1)(1-x)$, we get

$$\begin{aligned} 2x(1-x)+(4x-1)(x-1)+(x-1)(1-x)&=0,\\ 2x-2x^2+4x^2-5x+1-x^2+2x-1&=0,\\ x^2-x&=0. \end{aligned}$$

This equation is satisfied when $x=0$ and when $x=1$.

If we put 0 for x the original equation is satisfied. But if we put 1 for x the left member has no value. Hence 1 is not a root of this equation, although it is a root of the equation obtained by clearing the original equation of fractions. See the note at the end of § 64.

This question will be more carefully discussed in the Second Course. It is mentioned here in order to impress upon the student the necessity of checking the solutions of all fractional equations containing the unknown in a denominator.

EXERCISES

Solve the following equations and check each solution:

1. $\dfrac{x+4}{x-4}=5.$

2. $\dfrac{y-1}{2y-5}=\dfrac{y-5}{2y-2}.$

3. $\dfrac{3-x}{3-5}\div\dfrac{2-x}{2-5}=-1.$

4. $\dfrac{y-2}{2y-4}=\dfrac{y-4}{y-1}+\dfrac{1}{4}.$

5. $\dfrac{5}{1-x}=\dfrac{3}{6-x}.$

6. $\dfrac{x+7}{x-2}=\dfrac{x+1}{x-3}.$

7. $\dfrac{1}{x+3}+\dfrac{2}{x-3}=\dfrac{3}{9-x^2}.$

8. $\dfrac{4}{x-3}+\dfrac{2x+3}{6}=\dfrac{x}{3}.$

9. $\dfrac{6x^2+2x+2}{5x^2-x+7}=\dfrac{6}{5}.$

10. $\dfrac{2}{2x-4}+\dfrac{3}{5x-10}=4.$

11. $\dfrac{x-5}{x+4}-\dfrac{3}{2}=\dfrac{2-x}{3x+12}.$

12. $\dfrac{x+7}{x-7}=\dfrac{8-x^2}{49-x^2}+\dfrac{1}{x^2-49}.$

13. $\dfrac{x-2}{x-3}+\dfrac{x+1}{x+2}=\dfrac{2x^2-7x+2}{x^2-x-6}.$

14. $\dfrac{y+5}{y-5}+\dfrac{3y-2}{y+4}=\dfrac{4y^2+3y-3}{y^2-y-20}.$

15. $.2x + .07 = .06x - .14.$

HINT. In solving such an equation it is best first to make all the numerical coefficients integers by multiplying both members of the equation by some power of 10. In this case 100 ($= 10^2$) should be chosen.

16. $.9 + .45x + .063 = 7.13 - .431x.$

17. $.4(3x - .05) - 2.5(3.5x + 1) + 10.07 = 0.$

18. $$\frac{.45x}{.28} - \frac{1.16x}{.42} = \frac{9.7}{4.2}.$$

First express each fraction without decimals in the denominator and then proceed in the usual way.

19. $$\frac{2.5}{4.6x} - \frac{4.1}{21.3x} = 7.6.$$

20. $$d' = \frac{d}{1 + Dt}.$$ (Solve for t.)

This formula occurs in certain problems concerning expansion by heat.

21. $$h = \frac{5508H}{5508 + t}.$$ (Solve for t.)

Barometer.

22. $$C = \frac{E}{R + r}.$$ (Solve for R.)

Electricity.

23. $$C = \frac{mE}{R + mr}.$$ (Solve for r.)

Electricity.

24. $$C = \frac{E}{\frac{1}{2}R + r + l}.$$ (Solve for l.)

Electricity.

25. $\frac{1}{F}=\frac{1}{f_1}+\frac{1}{f_2}$. (Solve for F.)

26. $\frac{1}{R}=\frac{1}{r_1}+\frac{1}{r_2}$. (Solve for R.)

The student should note that equations (25) and (26) are essentially the same, the difference being merely in the names of the numbers involved. The first occurs in the theory of lenses and the second in the theory of electricity.

27. $\frac{1}{f}=\frac{m-1}{m}\left(\frac{1}{r}+\frac{1}{s}\right)$. (Solve for f.)

Lenses.

28. $\frac{1}{D}-\frac{1}{d}=\frac{1}{f}$. (Solve for d.)

Microscope.

29. $\frac{1}{10}-\frac{1}{p}=\frac{1}{x}$. (Solve for x.)

Spectacles.

30. $\frac{1}{F+f}+\frac{1}{a}=\frac{1}{f}$. (Solve for a.)

Telescope.

31. $\frac{1}{f}=\frac{1}{f_1}+\frac{1}{f_2}-\frac{D}{f_1f_2}$. (Solve for f_2.)

Systems of lenses.

32. $V=\frac{Q}{R}-\frac{Q}{R'}$. (Solve for R.)

Electricity.

33. $a=\frac{w}{x}s(t-h)-(h-T)$. (Solve for x.)

Theory of heat.

34. $Q = kA\frac{v_2 - v_1}{x}t.$ (Solve for x.)

Theory of heat.

35. $P = \frac{.31\ V(H - \frac{3}{8}FE)}{760(1 + at)}.$ (Solve for t.)

Gases and vapors.

36. $\frac{w}{d}(1 + mt) = \frac{W}{D}x + \frac{M}{d}mt - \left(\frac{W}{D} + \frac{M}{d}\right)Kt.$ (Solve for x.)

Expansion of metals.

37. $\frac{x - a}{x - b} \cdot \frac{c - b}{c - a} = -2.$ (Solve for x.)

PROBLEMS

1. Divide 75 into two parts whose quotient is $\frac{2}{3}$.

2. What positive number is 2 more than three times its reciprocal?

3. Two numbers differ by 4 and the reciprocal of the larger one is two thirds of the reciprocal of the smaller one. What are the numbers?

4. The difference between two numbers is 20 and their quotient is $\frac{2}{7}$. What are the numbers?

5. The difference between two numbers is D and their quotient is Q. Express the two numbers in terms of D and Q.

Solve Problem 4 by substituting in these formulæ.

6. If the abscissæ of the points A, B, and C are 2, 3, and 5 respectively, what must be the abscissa of D in order that $\frac{DA}{DB} = 4\frac{CA}{CB}$?

7. The difference between the speeds of two ships is 5 knots per hour. The faster ship goes 150 knots while the slower one is going 120 knots. Find the speeds.

8. A French physicist found that when a bell was struck by a hammer at one end of an iron tube 3120 feet long two sounds were heard at the other end 2 seconds apart. The first sound was transmitted by the tube itself, and the other by the inclosed air. What is the velocity of sound in iron as shown by this experiment? Take the velocity of sound in air to be 1100 feet a second.

9. Two automobiles started at the same time for a destination 50 miles away. One of them traveled 20 miles an hour and reached the destination 50 minutes ahead of the other one. What was the rate of the second one?

10. What number must be subtracted from each term of the fraction $\frac{13}{15}$ in order that the resulting fraction shall equal $\frac{1}{3}$?

11. A can do a piece of work in 5 days and B can do it in 8 days. How long will it take the two together to do it?

Let $x =$ the number of days it will take the two to do it. Then $\frac{1}{x} =$ the part of the whole the two can do in 1 day. Now A can do $\frac{1}{5}$ of the whole in 1 day and B can do $\frac{1}{8}$ of it in 1 day. They therefore can do together $\frac{1}{5}+\frac{1}{8}$ of it in 1 day. Hence

$$\frac{1}{x}=\frac{1}{5}+\frac{1}{8}.$$

12. A and B together finish a piece of work in 10 days. A could have done it alone in 16 days. How long would it have taken B alone to do it?

13. A can do a piece of work in 16 days. But after he has worked alone for 3 days he is joined by B and the two together finish it in 4 days more. How long would it have taken B to do it alone?

14. A boy who can mow his lawn in 90 minutes works 10 minutes at it and is then joined by a second boy with his own mower. Working together they finish the lawn in 40 minutes more. Which boy works the faster?

15. A can do a piece of work in a days and B can do it in two thirds of this time. How long will it take the two together to do it?

16. Two men, A and B, working together complete a given job in 8 days. They noticed that A did a third of the whole job while B was doing a fourth of it. How long would it have taken A alone to complete the job?

17. A was working on a job that would take him 10 days if he worked alone. But after he had been working 3 days he was joined by B and the two worked together for 2 days. Then A dropped out and B finished in another day. How long would it have taken B to do it alone?

18. It takes A, B, and C 10, 12, and 15 days respectively to do a certain amount of work. How long would it take if they all worked together?

19. A part of $4500 is to be invested at 6% and the remainder at 4%. How much should each investment be in order that the two annual incomes be the same?

20. The sum of $7600 is invested in 2 enterprises at 6% and 5%, respectively. The total income is $410. How was the division made?

21. The circumference of the rear wheel of a carriage is 14 feet. What should be the circumference of the front wheel in order that it may make 60 revolutions more in a mile than the rear wheel?

22. What number must be subtracted from each of the numbers 15, 20, 21, and 24 in order that the quotient of the first two remainders shall equal the quotient of the last two?

23. What number must be subtracted from 7 and 8 in order that the quotient of the remainders equal the quotient of the remainders obtained by subtracting twice this number from 12 and 13?

24. Two automobiles start from A for B at the same time. One of them, which goes 5 miles per hour faster than the other one, reaches B, 144 miles from A, and at once returns towards A, meeting the slower machine 16 miles out from B. What is the rate of each?

The student will find it advantageous to arrange the data of such problems in the form of a table. Thus, if x represents the rate of the slower machine:

	DISTANCE	RATE	TIME
Slower machine	$144 - 16$	x	$\frac{128}{x}$
Faster machine	$144 + 16$	$x + 5$	$\frac{160}{x+5}$

From this table we see immediately, since the two machines travel the same number of hours, that

$$\frac{128}{x} = \frac{160}{x+5}.$$

25. Two bicyclists start from M for N at the same time. One of them, who goes 3 miles an hour faster than the other, reaches N 60 miles from M and at once returns towards M, meeting the other one 12 miles out from N. What is the rate of each?

26. A man traveled from M to N, a distance of 210 miles, at a uniform rate. At the end of the first 130 miles he stopped for 3 hours and then resumed his journey at the same rate. When he reached N he found that his time to the stopping place was $\frac{8}{15}$ of the total time. What was his total time and his rate?

27. A tank is supplied by two pipes. It is found that the first can fill it in 6 hours and the second in 8 hours. How long would it take to fill the tank if both pipes were open?

28. How long will it take to fill this tank if the second pipe is an outlet pipe?

29. It is found by trial that one pipe can fill a cistern in 12 hours, and that another one can fill it in 10 hours. When these two pipes and an outlet pipe were all running the cistern was filled in 17 hours. How long would it take the outlet pipe to empty the cistern?

30. An aviator made 30 miles an hour against the wind on the first part of a journey of 60 miles. After he had gone 25 miles the wind decreased one half. The entire time for the trip was $1\frac{5}{6}$ hours. What was the speed of the wind?

The speed against the wind equals the speed in calm weather minus the speed of the wind, and the speed with the wind equals the speed in calm weather plus the speed of the wind.

31. A boat crew that can row at the rate of 8 miles an hour finds that it takes twice as long to go 5 miles upstream as it does to come down. What is the rate of the current?

REVIEW OF CHAPTERS VIII–XII

1. Describe a check for the work of dividing a polynomial by a polynomial.

2. Does it make any difference in the result of dividing a polynomial by a polynomial whether the dividend and the divisor are arranged according to the descending or according to the ascending powers of some letter?

3. Define the highest common factor and the lowest common multiple of two or more polynomials.

4. Is the H. C. F. of two or more polynomials a divisor of the L. C. M. of these polynomials?

5. Give the rule for finding the H. C. F. of two or more polynomials.

6. Give the rule for finding the L. C. M. of two or more polynomials.

7. Find the H. C. F. and the L. C. M. of $2a^2-5ab-3b^2$ and $3b^2+5ab-2a^2$.

8. On what principle does the rule for reducing a fraction to its lowest terms rest?

9. Reduce $\frac{5m^2+16my+12y^2}{5m^2+4my-12y^2}$ to its lowest terms.

10. Name the different types of polynomials you know how to factor.

11. Factor a^4-ab^{12}.

12. Factor $m^4-m^2x^2+16x^4$.

13. Factor $a^2+8ab+16b^2-c^2$.

14. Factor $acx^2+(bc+ad)xy+bdy^2$.

15. Factor a^8-m^8.

16. Describe some changes that can be made in the two members of an equation without affecting the number described by the equation, but that cannot be made in the numerator and the denominator of a fraction without changing the value of the fraction.

17. Mention some changes that can be made in the numerator and the denominator of a fraction without altering its value.

18. Simplify $\dfrac{\frac{1}{x}}{\frac{1}{x}+\frac{1}{y}}+\dfrac{\frac{1}{y}}{\frac{1}{x}-\frac{1}{y}}$.

19. Simplify $\frac{1}{x+1}+\frac{1}{x+2}+\frac{1}{x+3}$.

20. Simplify $\left(\frac{1}{r_1}-\frac{1}{r_2}\right)\left(\frac{1}{r_1^2}+\frac{1}{r_1r_2}+\frac{1}{r_2^2}\right)$.

21. Reduce $\frac{3x^4-2x^3-5x^2+3x+4}{x^2+x+3}$ to a mixed expression.

22. Simplify $\frac{\frac{1}{x_1}-\frac{1}{x_2}}{\frac{1}{x_1x_2}}$.

23. Simplify $\frac{\frac{1}{x_1}-\frac{1}{x_2}}{\frac{1}{x_1x_2}} \div \frac{\frac{1}{x_1}+\frac{1}{x_2}}{\frac{1}{x_1x_2}}$.

24. Solve the equation $D=\frac{760\,p(l+at)}{.31\,v(H-h)}$ for H.

25. Solve the equation $d'=\frac{d}{1+Dt}$ for D.

26. Solve the equation $\frac{x-1}{x-2}=\frac{3}{2}$.

27. Given the points A, B, and C with the abscissæ a, b, and c respectively, find the abscissa of a fourth point D such that $\frac{AD}{BD}=\frac{AC}{BC}$.

28. What must be the abscissa of D in order that $\frac{DA}{DB}=4\frac{CB}{CA}$?

29. The difference in the speeds of two ships is 4 knots per hour. What must be their speeds in order that the faster one may go 200 knots while the other is going 160 knots?

30. If A can do a piece of work in a days and B can do it in b days, in how many days can the two together do it?

CHAPTER XIII

SQUARE ROOT AND RADICALS

129. There was described in § 95 a method for writing down the square of a binomial, and an application of the simple rules of multiplication would enable us to find the square of any polynomial. Now there are occasions upon which it is desirable to reverse this process; that is, to find a polynomial whose square is a given polynomial.

Definition. — Such a polynomial is called a **square root** of the given polynomial.

130. The problem of finding a square root of a polynomial may be considered a problem in division with both divisor and quotient unknown. We have to determine the divisor in such a way that it shall be the same as the quotient.

Not every polynomial is the square of a polynomial and therefore not every polynomial has a square root that can be written as a polynomial.

Definition. — A polynomial that has a polynomial square root is said to be a **perfect square.**

The problem before us is to devise a means for finding the square roots of polynomials that are perfect squares.

131. Consider first a trinomial square. The trinomial $a^2 + 2\,ab + b^2$ is typical of all such trinomials. We already know that one of its square roots is $a + b$ (see § 95), and hence a detailed consideration of this case for its own sake is unnecessary. It is, however, worth while for the sake of the light it throws on the process of finding the square roots of polynomials other than trinomials.

The first term of the square root is a square root of the first term of the trinomial, it being assumed that the trinomial has been arranged according to the descending or the ascending powers of one of its letters. If we subtract from the trinomial the square of the first term of the root the difference is $2\,ab+b^2$; and the second term of the root is the quotient obtained by dividing the first term of this remainder by twice the first term of the root.

Suppose, for example, we want to find a square root of $36\,x^2+84\,xy+49\,y^2$.

We observe first that this is already arranged according to the descending powers of x, and that a square root of the first term is $6\,x$.

$$\begin{array}{rl}
 & 36\,x^2 + 84\,xy + 49\,y^2 \,\underline{|\,6\,x+7\,y} \\
 & \underline{36\,x^2\qquad\qquad\qquad\quad} \\
12\,x + 7\,y & |\,84\,xy + 49\,y^2 \\
 & \underline{|\,84\,xy + 49\,y^2}
\end{array}$$

We subtract the square of $6\,x$ from the trinomial and divide the first term of the remainder, $84\,xy+49\,y^2$, by $2\cdot 6\,x = 12\,x$. The quotient is $7\,y$. We add the $7\,y$ to the $12\,x$ and multiply the sum by $7\,y$. When we subtract the product from the first remainder the final remainder is zero. This shows that the remainder is zero when we subtract from the polynomial

$$(6\,x)^2 + 7\,y(2\cdot 6\,x + 7\,y) = 36\,x^2 + 2\cdot 6\,x\cdot 7\,y + (7\,y)^2.$$

This subtraction was performed in two steps, but the result is the same as if it had all been done at one time. What we have subtracted is equal to the square of the first term of the root, plus twice the product of the first and second terms, plus the square of the second term — that is, is equal to the square of the root.

If the final remainder is not zero, the given trinomial is not a perfect square.

132. Every perfect square has two square roots, since the square of any polynomial is, in consequence of the rule of signs in multiplication, the same as the square of this polynomial with the signs of all its terms changed. It is not necessary to employ the process just described to find both of the square roots of a perfect square. When we have found one of them in this way, the other can be found by changing the signs of all the terms of the first one. Thus $-6x-7y$ is also a square root of $36x^2+84xy+49y^2$.

EXERCISES

Find the square roots of each of the following trinomials or show that it is not a perfect square:

1. $4a^2+12ab+9b^2$.
2. $9x^2-30xy+25y^2$.
3. $25b^2+64c^2-80bc$.
4. $14mn+49m^2+2n^2$.
5. $9a^2-12ab+3b^2$.
6. $x^4+x^2y^2+y^4$.
7. $x^4-x^2y^2+y^4$.
8. $x^2+6xy+9y^2$.
9. $144a^2-24ax+x^2$.
10. $9+81b^2+36b$.
11. $1-12x+36x^2$.
12. $(x-y)^2+6z(x-y)+9z^2$.
13. $a^2+2a(b+c)+(b+c)^2$.
14. $25-20x+4x^2$.
15. $2r^2+12rs+9s^2$.
16. $\frac{1}{4}m^2+mn+n^2$.
17. $a^2b^2+2abcd+c^2d^2$.
18. $(x^2+y^2)^2-x^2y^2$.
19. $2ab-a^2-b^2$.
20. $\frac{4}{9}b^2c^2+2xybc+\frac{9}{4}x^2y^2$.
21. $60ax+25a^2+36x^2$.
22. $4m^2+6mn+9n^2$.
23. $9a^2b^2x^2-30abxy+25y^2$.
24. $9a^2b^2+30abxy+25y^2x^2$.
25. $a^{2m}+2a^mb^n+b^{2n}$.
26. $a^{2x}b^{2y}-6a^xb^y+9$.
27. $a^{4x}-a^{2x}+1$.
28. $(a+2b)^2-8(a+2b)+16$.
29. $x^2-10(y-2z)+25(y-2z)^2$.
30. Can you write down a binomial that is a perfect square?

133. This process can easily be adapted to finding square roots of perfect squares other than trinomials. How this can be done is illustrated here for those polynomials whose square roots are trinomials.

Example. — Find the square roots of

$$24x^2 - 32x + 16 + x^4 - 8x^3.$$

Arranging this in the descending powers of x, we get

$$\begin{array}{lrrl}
 & & x^4-8x^3+24x^2-32x+16 & \underline{|\,x^2-4x+4} \\
 & (x^2)^2 & \underline{x^4\qquad\qquad\qquad\qquad} & \\
\text{1st trial divisor,} & 2\cdot x^2=2x^2 & -8x^3+24x^2-32x+16 & \text{1st remainder} \\
\text{1st complete divisor,} & 2x^2-4x & \underline{-8x^3+16x^2\qquad\qquad\quad} & (2x^2-4x)(-4x) \\
\text{2d trial divisor,} & 2(x^2-4x)=2x^2-8x & 8x^2-32x+16 & \text{2d remainder} \\
\text{2d complete divisor,} & 2x^2-8x+4 & \underline{8x^2-32x+16} & (2x^2-8x+4)\,4 \\
 & & 0 & \text{Final remainder}
\end{array}$$

The required square roots are therefore

$$x^2 - 4x + 4 \quad \text{and} \quad -x^2 + 4x - 4.$$

The first term of the root is a square root of the first term of the given polynomial, after it has been properly arranged. The second term of the root is the quotient obtained by dividing the first term of the first remainder by the first trial divisor. The third term of the root is the quotient obtained by dividing the first term of the second remainder by the first term of the second trial divisor.

The general process is described in the following

Rule. — *In order to find a square root of a polynomial that is a perfect square, arrange the polynomial according to the descending or the ascending powers of one of its letters.*

Take a square root of the first term of the polynomial as the first term of the root, and subtract its square from the given polynomial.

Form a first trial divisor by doubling the part of the root already found. Divide the first term of the remainder by this trial divisor and take the quotient as the second term of the root.

Form a complete divisor by adding the quotient just found to the trial divisor. Multiply the complete divisor by the last term of the root and subtract the product from the first remainder.

If this second remainder is not zero, double the part of the root already found for a second trial divisor, and divide the first term *of the remainder by the first term of this trial divisor. Write the quotient as the next term of the root and also add it to the last trial divisor to form a complete divisor. Multiply this complete divisor by the last term of the root and subtract the product from the last remainder.*

Continue this process until a zero remainder is obtained.

EXERCISES

Find the square roots of each of the following polynomials:

1. $a^6 + 4a^5 + 4a^4 + 2a^3 + 4a^2 + 1$.
2. $18b + b^4 + 81 - 2b^3 - 17b^2$.
3. $36y^4 + 1 - 12y^3 - 11y^2 + 2y$.
4. $3x^4 + x^8 + 1 + 2x^6 + 2x^2$.
5. $16c^4 + 88c^3 + 97c^2 - 66c + 9$.
6. $13z^4 + 9z^8 + 36 + 42z^6 - 84z^2$.
7. $x^6 + 2x^5y - 2x^4 + x^4y^2 - 4x^3y - 2x^2y^2 + x^2 + 2xy + y^2$.
8. $400 + a^2 + 9a^4 - 66a^3 + 440a$.
9. $x^2 + y^2 + z^2 + 2xy + 2yz + 2zx$.
10. $6x^6 - 11x^4 + 9x^8 - 4x^2 + 4$.
11. $a^4 + 24a^3 + 136a^2 - 96a + 16$.
12. $144 - 48h - 92h^2 + 16h^4 + 16h^3$.
13. $64 - 64x + 48x^2 + x^6 - 4x^5 + 12x^4 - 32x^3$.
14. $-6m + 1 - 20m^3 + 15m^4 + m^6 - 6m^5 + 15m^2$.
15. $297x^2y^2 - 90x^3y - 360xy^3 + 144y^4 + 9x^4$.
16. $a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$.
17. $x^2 + 4y^2 + 9z^2 - 4xy + 6xz - 12yz$.
18. $x^2 + 4y^2 + 9z^2 + 4xy - 6xz - 12yz$.
19. $\frac{1}{4}a^2 + \frac{1}{3}ab + \frac{1}{6}bc + \frac{1}{4}ac + \frac{1}{16}c^2 + \frac{1}{9}b^2$.
20. $y^4 + 6y^3 + 11y^2 + 6y + 1$.

21. $m^4 - 4m^3n + 6m^2n^2 - 4mn^3 + n^4$.

22. $a^4 - 12a^3 + 36a^2 - 108a + 31$.

23. $4a^2 + 9x^2 + y^2 - 12ax + 4ay - 6xy$.

24. $x^6 - 12x + 13x^2 - 6x^4 + 4$.

25. $24a^2b^2 - 8a^2c^2 - 6b^2c^2 + 16a^4 + 9b^4 + c^4$.

26. $1 + 2ab - 2bc - 2ab^2c + b^2c^2 + a^2b^2$.

27. $25a^{2x} + 4b^{2y} + c^{2z} + 20a^xb^y + 10a^xc^z + 4b^yc^z$.

28. $x^{4m} + y^{4n} + z^{4n} + 2x^{2m}y^{2n} + 2y^{2n}z^{2n} + 2x^{2m}z^{2n}$.

29. $(a + b)^2 + c^2 + d^2 + 2d(a + b) + 2c(a + b) + 2cd$.

30. $(x + y)^4 - 4(x + y)^3 + 6(x + y)^2 - 4(x + y) + 1$.

SQUARE ROOTS OF NUMBERS

134. Every integer can be written as a polynomial. Thus, $59049 = 5 \cdot 10^4 + 9 \cdot 10^3 + 4 \cdot 10 + 9$. This fact enables us to apply the preceding rule to finding the square roots of numbers. But there is one important difference between the writing of numbers in our usual Arabic notation and the writing of polynomials. This difference is illustrated by the following example:

$$\overline{43}^2 = (4 \cdot 10 + 3)^2 = 16 \cdot 10^2 + 24 \cdot 10 + 9 = 18 \cdot 10^2 + 4 \cdot 10 + 9 = 1849.$$

Here in the simplification we have replaced $24 \cdot 10$ by $2 \cdot 10^2 + 4 \cdot 10$, whereas, in the polynomial $16t^2+24t+9$, we cannot say that $24t=2t^2+4t$. It is only when $t = 10$ or 0 that we can do this.

In formulating our rule for extracting the square root of a number, we must take this difference into consideration.

Now $10^2 = 100$, $10^3 = 1000$, $10^4 = 10000$, and so on.

Hence a square root of an integer less than 100 is less than 10. Now no integer less than 100 contains more than two digits and an integer less than 10 contains only one digit. That is, a square root of a perfect square containing one or two digits contains only one digit. In a similar way we see that a square root of a square integer containing three or four digits contains two digits, and so on.

Hence the number of digits in a square root of a square integer is equal to the number of periods of two digits each contained in the number. These periods are counted off from right to left and the last period may contain only one digit.

Suppose we want to extract the square roots of 1849.

$$\begin{array}{r r l}
 & & a+b \\
 & 18'49 & \underline{|\,40+3=43} \\
2\cdot 40=80 & 16\,00 & \\
\underline{3} & |\,2\,49 & \\
83 & \underline{|\,2\,49} & \\
 & 0 &
\end{array}$$

Hence the square roots of 1849 are 43 and -43.

After having divided the number into periods of two digits each as just explained, we take the largest perfect square that does not exceed the left-hand period. The square root of this followed by as many zeros as there are remaining periods in the number is the first term of the square root.

In dividing any remainder by the corresponding trial divisor to get the next term of the root, we must remember that the complete divisor is always greater than this trial divisor. If we neglect to make due allowance for this fact the product of the complete divisor and the last term of the root will be greater than the last remainder. When this occurs we may know that we have made the last term of the root too large.

Extract the square roots of 59049.

$$\begin{array}{r r l}
 & 5'90'49 & \underline{|\,200+40+3=243} \\
2\cdot 200=400 & 4\,00\,00 & \\
\underline{40} & |\,1\,90\,49 & \\
440 & \underline{|\,1\,76\,00} & \\
2(200+40)=480 & |\,14\,49 & \\
\underline{3} & \underline{|\,14\,49} & \\
483 & 0 &
\end{array}$$

The square roots are 243 and -243.

After the student has acquired some facility in extracting roots, he will find it desirable to shorten this work in the way suggested by the following re-working of the preceding example:

```
       5′90′49|243
       4
   44 |1 90
      |1 76
  483  |14 49
       |14 49
            0
```

If numbers and polynomials were written in the same way we should expect the first digit of a root to be the square root of the numbers formed by the digits of the left-hand period.

135. Since the square of a decimal fraction contains twice as many decimal places as the fraction, a square root of a decimal fraction contains half as many decimal places as the decimal fraction itself. No fraction containing an odd number of decimal places is a perfect square. If the fraction is a perfect square we divide it into periods of two places each, counting from the decimal point and proceed as with integers.

Find the square roots of 1254.5764.

```
        12′54.57′64|35.42
         9
   65  |3 54
       |3 25
  704   |2957
        |2816
 7082    |14164
         |14164
              0
```

The square roots are 35.42 and – 35.42.

Find the square roots of .455625.

$$
\begin{array}{r|l}
 & .45'56'25 \;\big|\; .675 \\
 & 36 \\ \hline
127 & 956 \\
 & 889 \\ \hline
1345 & 6725 \\
 & 6725 \\ \hline
\end{array}
$$

The square roots are .675 and – .675.

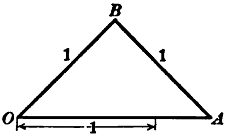
EXERCISES

Find the square roots of each of the following numbers:

1. 289.
2. 676.
3. 10404.
4. 18705625.
5. 62001.
6. 14641.
7. 841.
8. .01.
9. 9.4864.
10. .00001369.
11. 15625.
12. .9216.
13. 2.3104.
14. 1312.6129.
15. 1.2321.
16. .0001.
17. 948.64.
18. 173056.
19. 156.25.
20. .00003721.
21. 2777290000.
22. 1.7956.
23. 96.4324.
24. .138384.
25. 6130.89.
26. 385641.
27. 15876.
28. 225.
29. 729.
30. .00008649.
31. 1253.16.

136. An Addition to our List of Numbers.—In the discussion of negative numbers (Chapter III) we found that the points of a line could be designated with less ambiguity by the use of positive and negative numbers than by the use of the numbers of arithmetic. The only numbers we have so far considered are the rational numbers, that is, positive and negative integers and the quotients of such integers. But we cannot represent all the points of a line by these numbers. The following illustration will make this evident:

On the axis with the origin at O, suppose we take OA as the hypotenuse of a right triangle OBA, each leg of which is of length 1. Now we know from geometry (see p. 95) that $\overline{OA}^2 = \overline{OB}^2 + \overline{BA}^2$. Hence $\overline{OA}^2 = 1^2 + 1^2 = 2$.



Hence A is a point on the axis whose distance from the origin is such that its square is 2. But there are no integers a and b such that $\left(\frac{a}{b}\right)^2=2$, as will be proved in the Second Course. Therefore we cannot represent the point A by any of the numbers of the kind we have been using and if we want to have every point on the line represented by a number we must bring into use some new numbers.

Since the number that is to represent A must have its square equal to 2, we shall call it a *square root of 2* and represent it by the symbol $\sqrt{2}$.

137. This number, $\sqrt{2}$, is only a single representative of a whole group of numbers that we must take into account if we want a number for every point of our axis. The numbers $\sqrt{5}$, $\sqrt[3]{7}$, $\sqrt[6]{20}$, and π (the ratio of the circumference of a circle to its diameter) are examples of these new numbers.

Definitions. — In general we represent a number whose nth power is a by the symbol $\sqrt[n]{a}$. We call this number an **nth root of a** and $\sqrt[n]{\ }$ a **radical sign**, with the **index n**. When the index is 2, it is omitted. Thus $\sqrt{5}$ represents a number whose square is 5, or a square root of 5.

Expressions of the form $b\sqrt[n]{a}$ are called **radicals.**

These new numbers are called **irrational numbers.**

138. Suppose we take an integer that is not a perfect square, such as 124, and apply to it the preceding rule for getting the square root of a perfect square, forming as many periods as we wish of two zeros each to the right of the decimal point.

	1'24.00'00'00	11.135
	1	
21	24	
	21	
221	300	
	221	
2223	7900	
	6669	
22265	123100	
	111325	

	Result	Square of Result
The result obtained without adding any periods is	11	121
The result obtained by adding one period is	11.1	123.21
The result obtained by adding two periods is	11.13	123.8769
The result obtained by adding three periods is	11.135	123.988225

It will be observed that the more periods of zeros we add to the right of the decimal point, the more nearly the square of the result equals 124, that is, the more nearly our result equals $\sqrt{124}$. But no matter how far we carry this process we will never get a result whose square is exactly 124, since 124 is not a perfect square, and $\sqrt{124}$ is accordingly one of the new kind of numbers. The successive results are called the **approximate values** of $\sqrt{124}$. We say that the approximate value to two decimal places of $\sqrt{124}$ is 11.14, since this is nearer to the correct value than 11.13. The student should note that in order to get the approximate value of a number to a given number of decimal places we must compute one more decimal place than this required number.

Note. — The student is cautioned against the idea that these irrational numbers are not definite, precise values. They are as definite and precise as the rational numbers since they represent definite points on a line.

139. Definition. — A **surd** is an irrational number some integral power of which is a rational number.

For example, $\sqrt[5]{10}$ is a surd since it is irrational and its fifth power equals 10. The number π is irrational but is not a surd since no power of it is a rational number.

EXERCISES

Find the approximate value to two decimal places of each of the following surds:

1. $\sqrt{2}$.
2. $\sqrt{3}$.
3. $\sqrt{28}$.
4. $\sqrt{1.1}$.
5. $\sqrt{7}$.
6. $\sqrt{8}$.
7. $\sqrt{120}$.
8. $\sqrt{2.51}$.
9. $\sqrt{7.6}$.

10. $\sqrt{45.2}$.

11. $\sqrt{.6666}$.

12. $\sqrt{.1}$.

13. $\sqrt{.03}$.

14. $\sqrt{.125}$.

15. $\sqrt{.007}$.

16. $\sqrt{3.162}$.

17. $\sqrt{.004}$.

18. $\sqrt{.9}$.

19. $\sqrt{.025}$.

20. $\sqrt{.016}$.

21. $\sqrt{.2}$.

22. $\sqrt{5}$.

23. $\sqrt{50}$.

24. $\sqrt{.5}$.

25. $\sqrt{110}$.

26. $\sqrt{.009}$.

27. $\sqrt{125}$.

28. $\sqrt{160}$.

29. $\sqrt{1.6}$.

30. $\sqrt{4.9}$.

140. A square root of a common fraction is a fraction whose numerator is a square root of the given numerator and whose denominator is a square root of the given denominator. But it is not always best to find the square root of a fraction in this way. For example, if we proceeded in this way to get the approximate value of $\sqrt{\frac{5}{6}}$ we should get $\sqrt{\frac{5}{6}} = \frac{2.236}{2.449}$. Now in order to express this as a decimal we must divide by a fairly long decimal. Moreover, the result will not be a good approximation since both the numerator and the denominator of the last fraction are given only approximately. It would be better either to reduce $\frac{5}{6}$ to a decimal first and then get an approximate square root of the decimal, or to reduce $\frac{5}{6}$ to a fraction whose denominator is a perfect square.

Thus $\frac{5}{6} = \frac{30}{36}$, and $\sqrt{\frac{30}{36}} = \frac{\sqrt{30}}{6} = \frac{5.477}{6} = .913$. Following the former plan, we have $\frac{5}{6} = .833333$, and $\sqrt{\frac{5}{6}} = \sqrt{.833333} = .913$.

Is it always possible to reduce a fraction to an equal fraction whose denominator is a perfect square?

EXERCISES

Find the approximate value to two decimal places of each of the following surds:

1. $\sqrt{\frac{1}{2}}$. 2. $\sqrt{\frac{1}{3}}$. 3. $\sqrt{\frac{1}{5}}$. 4. $\sqrt{\frac{1}{7}}$. 5. $\sqrt{\frac{9}{10}}$. 6. $\sqrt{\frac{3}{4}}$.

7. $\dfrac{2}{\sqrt{5}}$.

8. $\dfrac{5}{\sqrt{11}}$.

9. $\dfrac{8}{\sqrt{15}}$.

10. $\sqrt{\frac{3}{2}}$.

11. $\sqrt{\frac{2}{5}}$.

12. $\sqrt{\frac{5}{7}}$.

13. $\dfrac{1}{\sqrt{10}}$.

14. $\sqrt{\frac{4}{3}}$.

15. $\sqrt{\frac{2}{3}}$.

16. $\dfrac{\sqrt{2}}{2}$.

17. $\dfrac{\sqrt{3}}{3}$.

18. $\sqrt{\frac{6}{5}}$.

19. $\sqrt{1\frac{1}{2}}$.

20. $\sqrt{3\frac{1}{4}}$.

21. $\sqrt{1\frac{2}{3}}$.

22. $\sqrt{2\frac{1}{5}}$.

141. Simplification of Radicals.—A radical of index 2 is said to be *simplified* when it has been reduced to an equal expression in which the part under the radical sign is an integer, or a polynomial with integral coefficients if it contains any literal terms, and contains no factors raised to powers equal to or greater than 2.

Thus $\sqrt{\frac{1}{2}}$, $\sqrt{\frac{1}{3}x - \frac{1}{4}y}$, $\sqrt{27}$ are not in their simplest forms.

If the part under the radical sign is fractional the radical can be changed into an equal radical in which the part under the radical sign is integral by the method described in § 140. If the part under the radical sign contains a factor raised to a power equal to or greater than 2, the radical can be simplified in the way illustrated by the following examples:

$$\sqrt{160} = \sqrt{4^2 \cdot 10} = \sqrt{4^2} \cdot \sqrt{10} = 4\sqrt{10};\ \sqrt{x^4 - x^3 - 2x^2} = \sqrt{x^2(x^2 - x - 2)}$$
$$= x\sqrt{x^2 - x - 2};$$

and

$$\sqrt{a^3b^5} = \sqrt{a^2b^4 \cdot ab} = ab^2\sqrt{ab}.$$

We here utilize the principle that $\sqrt{ab} = \sqrt{a} \cdot \sqrt{b}$.

EXERCISES

Compute the values of the following surds to two decimal places without extracting any roots, it being given that $\sqrt{2} = 1.414$, $\sqrt{3} = 1.732$, and $\sqrt{7} = 2.646$.

1. $\sqrt{27}$.
2. $\sqrt{\frac{1}{8}}$.
3. $\sqrt{50}$.
4. $\sqrt{14}$.
5. $\sqrt{42}$.
6. $\sqrt{48}$.
7. $\dfrac{1}{\sqrt{3}}$.
8. $\dfrac{1}{\sqrt{2}}$.
9. $\sqrt{175}$.
10. $\sqrt{8} + \sqrt{\frac{1}{2}}$.
11. $\sqrt{28} + \sqrt{63}$.
12. $\sqrt{108} + \sqrt{147} - 2\sqrt{3}$.
13. $\sqrt{50} + 4\sqrt{98} + \dfrac{3}{\sqrt{2}}$.
14. $\dfrac{2}{\sqrt{7}} - 3\sqrt{175}$.
15. $\sqrt{24} - \sqrt{216}$.
16. $-3\sqrt{56} + 5\sqrt{126} - 10\sqrt{\frac{2}{7}}$.

Simplify the following:

17. $\sqrt{98\, x^3y^4}$.
18. $6\sqrt{125\, ab}$.
19. $\sqrt{27\, a^3b^3c^3}$.
20. $\sqrt{81\, m^4n^2r^5}$.
21. $\sqrt{128\, x}$.
22. $\sqrt{17\, xyz}$.
23. $\sqrt{x^3 + 2\,x^2 - 4\,x - 8}$.

    Hint. $\sqrt{x^3 + 2\,x^2 - 4\,x - 8} = \sqrt{(x+2)^2(x-2)}$.

24. $\sqrt{(a-1)^3(a+1)^2}$.
25. $\sqrt{a^5 + 3\,a^4 + 2\,a^3}$.
26. $\sqrt{18\,x^2 + 36\,xy - 54\,y^2}$.

An algebraic expression containing radicals can sometimes be simplified by first simplifying the radicals contained in it.

For example, $\sqrt{27} - 3\sqrt{48} + 6\sqrt{75} = 3\sqrt{3} - 12\sqrt{3} + 30\sqrt{3} = 21\sqrt{3}$.

Also
$$\begin{aligned}&\sqrt{16\,ab^3} - 5\sqrt{a^3b} - b\sqrt{ab}\\ &= 4\,b\sqrt{ab} - 5\,a\sqrt{ab} - b\sqrt{ab}\\ &= (3\,b - 5\,a)\sqrt{ab}.\end{aligned}$$

Simplify each of the following expressions as far as possible:

27. $2\sqrt{40}-\sqrt{90}-2\sqrt{160}.$ **29.** $2\sqrt{\frac{5}{4}}-3\sqrt{\frac{5}{9}}+\sqrt{125}.$

28. $\sqrt{128}-\dfrac{2}{\sqrt{8}}+5\sqrt{2}.$ **30.** $\sqrt{112}+\sqrt{700}+\sqrt{343}.$

31. $\sqrt{a^2x+2\,abx+b^2x}+3\sqrt{a^2x}-5\sqrt{b^2x}.$

32. $\sqrt{\dfrac{8}{5}}-\dfrac{4\sqrt{90}}{3}+\sqrt{\dfrac{5}{2}}.$

33. $\sqrt{18\,x}+3\sqrt{27\,x}-2\sqrt{50\,x}-7\sqrt{48\,x}.$

34. $3\sqrt{a^3b^5}-5\sqrt{a^5b^3}.$

35. $\sqrt{(x^2-2\,xy+y^2)b}-5\sqrt{x^2b}-2\,y\sqrt{b}.$

36. $3\sqrt{a^3-a^2x}+5\sqrt{ax^2-x^3}+2\sqrt{(a+x)^2(a-x)}.$

37. $\sqrt{t+u}+\sqrt{t^3+t^2u}-\sqrt{36\,t+36\,u}-9\sqrt{tu^2+u^3}.$

38. $-3\sqrt{25\,a^3bc}+4\sqrt{49\,ab^3c}+8\sqrt{81\,abc^3}.$

39. $\sqrt{a^3+2\,a^2b+ab^2}-3\sqrt{a^3-2\,a^2b+ab^2}+5\sqrt{a^3}.$

40. $\sqrt{4\,x}+\sqrt{5\,y}+2\sqrt{xy^2}-7\sqrt{y^3}.$

41. $-7\sqrt{72}-8\sqrt{180}+5\sqrt{98}-10\sqrt{125}.$

APPLICATIONS OF SQUARE ROOT TO PROBLEMS OF GEOMETRY

142. Definitions and Theorems from Geometry. — The student is already familiar with the following definitions and theorems from geometry:

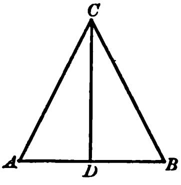
1. The area of a square is equal to the square of the length of one side.

2. The area of a triangle is equal to one half the product of its base and altitude.

3. The square of the hypotenuse of a right triangle is equal to the sum of the squares of the two sides.

4. The area of a circle is equal to the square of its radius multiplied by π ($= 3\frac{1}{7}$ approximately).

5. A line passing through the end of a radius and tangent to a circle is perpendicular to the radius.



6. A perpendicular drawn from a vertex of a triangle to the opposite side bisects this side if the other two sides of the triangle are equal to each other.

7. The sum of the lengths of the sides of a figure is called the *perimeter* of the figure.

8. A triangle is said to be *equilateral* when its sides are all of the same length.

9. A triangle is said to be inscribed in a circle, and the circle is said to be circumscribed about the triangle when the vertices of the triangle are on the circumference of the circle.

10. The altitude of an equilateral triangle inscribed in a circle of radius r is equal to $\frac{3}{2}r$.

PROBLEMS

Draw a figure for each of the following problems. Get all numerical results correct to two decimal places:

1. Find the hypotenuse of a right triangle whose two sides are 5 inches and 8 inches respectively.

2. If the hypotenuse is 8 inches and one of the sides is 5 inches, what is the other side?

3. If the hypotenuse is 12 feet and one of the sides is 6 feet, what is the other side?

4. Find the second side of a right triangle of which the hypotenuse is 10, and the other side 5.

5. Is there a right triangle whose sides are 6, 8, and 9 inches respectively?

6. The hypotenuse of a right triangle is 15 inches and the two sides are equal. How long are the sides?

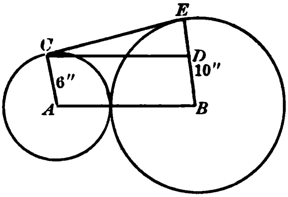
7. The hypotenuse of a right triangle is 16 and the two sides are equal. How long are these sides?

8. The hypotenuse of a right triangle is 20 and the two sides are equal. How long are the sides?

9. From a point 12 inches from the center of a circle whose radius is 5 inches, a tangent is drawn to the circle. How long is this tangent?

10. The length of the tangent from a point to a circle of radius 6 inches is 10 inches. How far is the point from the center of the circle?

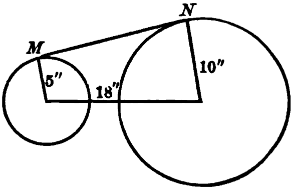
11. Two circles with radii of 6 inches and 10 inches respectively are tangent to each other. (See p. 95.) Find the length of their common tangent.



Hint. — Draw CD parallel to AB. Then $AB = CD$, and $AC = BD$. Find CE.

12. Two circles of radii 3 inches and 12 inches respectively are tangent to each other as in the figure of Ex. 11. Find the length of their common tangent.

13. The centers of two circles with radii of 5 inches and 10 inches respectively are 18 inches apart. Find the length of their common tangent.



14. Find the length of the line MN of the figure of Ex. 13 when the radii of the circles are 4 inches and 9 inches respectively and the centers are 20 inches apart.

15. The centers of two circles with radii of 3 inches and 1 inch respectively are 5 inches apart. At a point on the line connecting the centers 3.3 inches from the center of the larger circle a perpendicular to this line of centers is drawn. From

a point on the perpendicular 8 inches from where it cuts the line connecting the centers a tangent is drawn to each of the circles. Find the length of each of these tangents.

16. Find the altitude of an equilateral triangle each side of which is 12 inches.

17. Find the altitude of an equilateral triangle each side of which is 7 inches.

18. Find the altitude of an equilateral triangle whose perimeter is 20 inches.

19. What is the area of an equilateral triangle each side of which is 4 inches?

20. The perimeter of an equilateral triangle is 18 inches. Find its area.

21. What is the area of an equilateral triangle whose perimeter is 9 inches?

22. What is the area of an equilateral triangle each side of which is 8 inches?

23. An equilateral triangle contains 20 square inches. What is the length of each side?

Hint. — Let $2x =$ length of each side in inches. Then the altitude $= \sqrt{4x^2 - x^2} = \sqrt{3x^2} = x\sqrt{3}$, and the area $= x^2\sqrt{3}$. Hence $x^2\sqrt{3} = 20$ and $x^2 = \frac{20}{\sqrt{3}}$. Simplify this value and then get its square root.

24. What is the altitude of an equilateral triangle whose area is 12 square inches?

25. What is the radius of a circle whose area is 6 square feet?

26. What is the radius of a circle whose area is 9 square inches?

27. What is the circumference of a circle whose area is 20 square inches?

28. What is the circumference of a circle whose area is 2 square feet?

29. An equilateral triangle whose area is 12 square inches is inscribed in a circle. What is the area of the circle?

30. Find the circumference of the circle described in the preceding problem.

31. What is the perimeter of an equilateral triangle inscribed in a circle whose radius is 3 feet?

32. What is the area of the triangle described in the preceding problem?

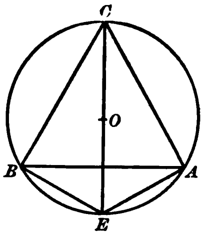
33. What is the area of the circle circumscribed about an equilateral triangle whose perimeter is 20 inches?

34. What is the circumference of a circle circumscribed about an equilateral triangle whose perimeter is 12 inches?

35. What is the area of an equilateral triangle inscribed in a circle of radius 6 inches?

36. What is the area of a circle circumscribed about an equilateral triangle whose altitude is 11 inches?

37. What is the area of a circle circumscribed about an equilateral triangle each of whose sides is 14 inches?



38. In the accompanying figure *ABC* is an equilateral triangle inscribed in a circle whose center is at *O*, and *COE* is a straight line perpendicular to the side *AB*. Find the area of the triangle *AEB* when the radius is 4 inches.

39. What is the relation between the areas of the triangles *ABC* and *ABE* of Problem 38?

40. Find the area of the triangle *ABE* of Problem 38 when the radius of the circle is r feet.

CHAPTER XIV

RATIO AND PROPORTION

143. Ratio. — The value of a fraction is called the **ratio** of the numerator to the denominator.

Thus the ratio of 6 to 2 is $\frac{6}{2} = 3$; the ratio of 4 to 5 is $\frac{4}{5}$.

The numerator of the fraction is called the **antecedent** of the ratio and the denominator is called the **consequent**.

Thus in the ratio $\dfrac{a}{b}$ the antecedent is a and the consequent is b.

The antecedent and the consequent are called the **terms** of the ratio.

Sometimes the ratio of a to b is indicated by the symbol $a : b$.

EXERCISES

If the weight of a given volume of a substance is s times the weight of the same volume of water, the substance is said to have the *specific gravity* s.

1. Find the specific gravity of each of the following substances:

SUBSTANCE	WEIGHT IN POUNDS OF 1 CU. FT.
Water	62.5
Copper	556.25
Silver	656.25
Pine	31.25
Ice	56.25
Marble	168.75

2. A circular plate 12 inches in diameter has a circumference of $37\frac{5}{7}$ inches, approximately. What is the ratio of the circumference to the diameter?

3. The circumference of a bicycle wheel 30 inches in diameter is found to be $94\frac{2}{7}$ inches, approximately. What is the ratio of the circumference to the diameter?

How does this result compare with that obtained in the preceding problem?

Arrange the following ratios in the descending order of magnitude:

4. $\frac{3}{5}, \frac{2}{3}, \frac{5}{7}$. 5. $\frac{8}{3}, \frac{9}{4}, \frac{1}{2}$. 6. $\frac{2}{3}, \frac{5}{9}, \frac{7}{8}$.

7. Write the ratio of $a^2 - b^2$ to $a^3 - b^3$ as a fraction reduced to its lowest terms.

Simplify the following ratios:

8. $\dfrac{x-2}{x-3} : \dfrac{x-2}{x^2-2x+3}$.

10. $\left(a^3 - \dfrac{b^4}{a}\right) : \left(\dfrac{a}{b} - \dfrac{b}{a}\right)$.

9. $(x^2 + xy + y^2) : (x^6 - y^6)$.

11. $\dfrac{a-b}{a^5-b^5}$.

12. Divide 70 into two parts that are in the ratio of 3 : 4.

13. Divide 42 into two parts that are in the ratio of 4 : 3.

14. Which is the greater ratio $\dfrac{x+3y}{x+2y}$ or $\dfrac{x+5y}{x+4y}$, it being understood that x and y are positive numbers?

15. If x is a positive number, which is greater, $\dfrac{2}{3}$ or $\dfrac{2+x}{3+x}$?

16. If x is a positive number, which is greater, $\dfrac{5}{4}$ or $\dfrac{5+x}{4+x}$?

17. Show that if a, b, and x are positive numbers and a is less than b, then $\dfrac{a}{b}$ is less than $\dfrac{a+x}{b+x}$.

18. Show that $\dfrac{a+x}{b+x}$ is less than $\dfrac{a}{b}$ when a, b, and x are positive numbers and a is greater than b.

Compare the statement bearing on this point in § 117.

19. What number must be added to the terms of the ratio $\frac{4}{9}$ to change its value to $\frac{2}{3}$?

20. What number must be subtracted from the terms of the ratio $\frac{17}{11}$ to change its value to 2?

21. The ratio of 4 to a certain number is equal to nine times the ratio of this number to 4. What is the number?

22. The ratio of 5 to a certain number is equal to three times the ratio of this number to 5. What is the number?

144. Proportion.—The statement that two fractions are equal is called a **proportion.**

Thus $\frac{2a}{3b} = \frac{5x}{2y}$, $\frac{a}{b} = \frac{c}{d}$, and $4 : 5 = 8 : 10$ are proportions.

When we are thinking of any of these equations, the first one, for example, as a proportion, we read it thus: $2a$ is to $3b$ as $5x$ is to $2y$.

In the proportion $\frac{a}{b} = \frac{c}{d}$, a, b, c, and d are called the **terms** of the proportion. The first and last terms are called the **extremes** and the second and third terms are called the **means.**

When we say that four numbers a, b, c, d are in proportion, we mean that the ratio of the first two equals the ratio of the last two; that is, $\frac{a}{b} = \frac{c}{d}$. This same relation is sometimes expressed by saying that a and b are *proportional* to c and d.

145. All the properties of a proportion are derived from the fact that a proportion is an equation. Thus, if we clear the equation

$$\frac{a}{b} = \frac{c}{d}$$

of fractions, we get

$$ad = bc.$$

Hence in any proportion the product of the means is equal to the product of the extremes.

Ex. Prove that if $ad = bc$, then $\frac{a}{b} = \frac{c}{d}$.

We can make use of this property of a proportion to get certain important information from the fact that four numbers are in proportion. Consider, for example, the proportion $\frac{3}{x} = \frac{7}{4}$. If we wish to know what number is described by this statement, we apply the preceding principle and find that

$$12 = 7x,$$

and that therefore $$x = \tfrac{12}{7}.$$

EXERCISES

Find the number described by each of the following proportions:

1. $\frac{x}{3} = \frac{5}{2}$.
2. $\frac{2}{y} = \frac{3}{4}$.
3. $\frac{25}{26} = \frac{R}{52}$.
4. $x : 5 = 3 : 8$.
5. $\frac{100}{6} : 9 = 30 : x$.
6. $\frac{x+1}{2x+1} = \frac{2}{3}$.
7. $10 : x = 2 : 7$.
8. $\frac{2x-3}{2x+3} = \frac{1}{2}$.
9. $\frac{x^2+x+1}{2x^2-3} = \frac{1}{2}$.
10. $\frac{x+1}{x-1} = \frac{x+4}{x+3}$.
11. $x : x + 5 = .849 : 1.698$.
12. $97 : x = 250 \cdot \frac{388}{3887} : 25 \times 1.366$.
13. $\frac{3}{\frac{2}{x}} = \frac{4}{9}$.
14. $\frac{r}{s} = \frac{t}{x}$. (Solve for x.)
15. $\frac{a+b}{a-b} = \frac{x}{x+1}$. (Solve for x.)
16. $\frac{h}{75} = \frac{1 + \frac{30}{5550}}{1 + \frac{15}{5550}}$.

17. $\dfrac{1+\dfrac{2}{x}}{1+\dfrac{1}{x}}=\dfrac{3}{2}.$

19. $\dfrac{t^2}{4}=\dfrac{144}{256}.$

18. $\dfrac{3-\dfrac{1}{x}}{1+\dfrac{2}{x}}=\dfrac{2}{3}.$

20. $\dfrac{x^2+1}{x^2+4}=\dfrac{5}{6}.$

146. Fourth Proportional. — If $\frac{a}{b}=\frac{c}{f}$, f is called the **fourth proportional** to a, b, and c.

If f is a fourth proportional to a, b, and c, then $f=\dfrac{bc}{a}$.

Thus, $\frac{2}{3}=\frac{12}{18}$ and therefore 18 is a fourth proportional to 2, 3, and 12.

147. Mean Proportional. — If $\frac{a}{m}=\frac{m}{b}$, m is called a **mean proportional** between a and b.

If m is a mean proportional between a and b, then $m^2=ab$, and $m=\pm\sqrt{ab}$. There are two mean proportionals between two given numbers.

For example, $\frac{2}{8}=\frac{8}{32}$ and $\frac{2}{-8}=\frac{-8}{32}$.

148. Third Proportional. — If $\frac{a}{b}=\frac{b}{t}$, t is called the **third proportional** to a and b.

If t is a third proportional to a and b, $t=\dfrac{b^2}{a}$.

Thus, $\frac{8}{12}=\frac{12}{18}$ and therefore 18 is the third proportional to 8 and 12; 18 is also the third proportional to 8 and -12.

EXERCISES

Find the mean proportionals between:

1. 1 and 16.
2. 3 and 12.
3. 5 and 125.
4. $\frac{1}{2}$ and $\frac{1}{8}$.
5. 5 and 7.
6. 4 and 25.
7. $\frac{1}{4}$ and $\frac{1}{9}$.
8. $x-y$ and $\dfrac{x-y}{4}$.

9. $a+b$ and $\dfrac{9}{a+b}$.

10. -3 and -5.

11. $\dfrac{a-b}{3}$ and $3(a-b)$.

12. -6 and -20.

13. Find the third proportional to the numbers in each of the preceding exercises.

Find the fourth proportional to each of the following sets of numbers:

14. 2, 3, and 4.

15. -3, 4, and 6.

16. $2\frac{1}{2}$, -5, and -7.

17. x, $x+1$, $x+2$.

18. a, $a+b$, $\dfrac{1}{a+b}$.

19. a^3, b^3, c^3.

20. $3a$, $2b$, and $9c$.

21. $x+y$, $x-y$, and x^2-y^2.

22. $\dfrac{r+s}{2}$, $\dfrac{2}{r-s}$, and r^2-s^2.

23. $a-b$, $\dfrac{1}{a+b}$, a^2-b^2.

24. $a-b$, a^2-b^2, $\dfrac{1}{a+b}$.

25. $\dfrac{1}{a+b}$, a^2-b^2, $a-b$.

PROBLEMS

1. The points A and B have the abscissæ 7 and 12 respectively. Find the abscissa of C if $\dfrac{AC}{CB}=\dfrac{2}{3}$.

2. The abscissæ of A and B being -4 and 2 respectively, find the abscissa of P which is such that $AP=2\,BP$.

3. Suppose that the abscissæ of A and B are 5 and -13 respectively. Find the abscissæ of the points of trisection of the segment AB.

4. The points P and Q have the abscissæ x_1 and x_2 respectively. Find the abscissa of the point R such that $\dfrac{PR}{RQ}=\dfrac{2}{3}$. Is there any other point S on the line such that $\dfrac{PS}{SQ}=\dfrac{2}{3}$?

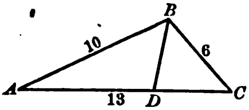
5. Find the abscissa of the point T such that $\frac{PT}{QT}=\frac{2}{3}$. How are R and T situated with respect to P and Q?

6. A piece of timber 15 feet long is to be cut into two parts the ratio of whose lengths is 4 : 5. What should be the lengths of the parts?

7. A beam 12 feet long is to be divided into two parts, the ratio of whose lengths is 2 : 3. How long should each of the parts be?

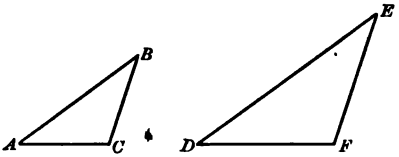
8. If $AB=10$, $BC=6$, $CA=13$, what are AD and DC, it being assumed that the angles ABD and CBD are equal? (See p. 122.)

9. What can you say about AD and DC in case $AB=BC$?



Facts from Geometry. — Two triangles are said to be *similar* if the three angles of one are equal respectively to the three angles of the other.

In two similar triangles the sides opposite equal angles are said to be corresponding sides. Thus in the similar triangles of the figure the angles at A and D are equal, and accordingly BC and EF are corresponding sides.



Corresponding sides of two similar triangles are proportional. Thus in the figure $\frac{AB}{DE}=\frac{BC}{EF}=\frac{CA}{FD}$.

10. The sides of a triangle are 8, 12, and 15 respectively. The shortest side of a similar triangle is 5. How long are the other sides?

11. How long are the sides of a triangle similar to that of Problem 8 if the sum of the sides is 30 inches?

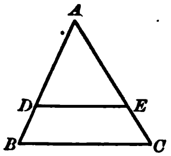
12. Two similar triangles have perimeters of 40 inches and 60 inches respectively. Two sides of the first triangle are 8 inches and 15 inches respectively. What are the lengths of all the other sides?

13. The sides of a triangle are 8 inches, 11 inches, and 15 inches respectively, and a similar triangle is to be constructed whose perimeter shall be 40 inches. What will be the lengths of its sides?

14. A tree casts a shadow on the level ground 40 feet long when a vertical rod 5 feet long casts a shadow 4 feet long. How high is the tree?

15. If a man 5 feet 6 inches tall casts a shadow 25 inches long, how high is a spire that casts a shadow 24 feet long?

Fact from Geometry. — It is proved in geometry that a line parallel to one side of a triangle divides the other two sides into four proportional parts and that the new triangle formed is similar to the original triangle.



Thus in the figure $\frac{AD}{DB} = \frac{AE}{EC}$.

16. If $AB = 20$, $AC = 18$, and $AD = 14$, how long are AE and EC?

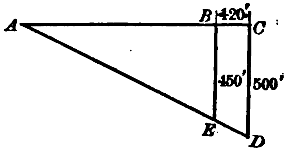
Hint. — Let $x =$ length of AE.

Then $18 - x =$ length of EC.

Hence $\frac{14}{6} = \frac{x}{18 - x}$.

17. What is AD, if $DB = 8$ and $EC = \frac{AC}{3}$?

18. What is AD, if $BC = b_1$, $DE = b_2$, and $DB = a$?

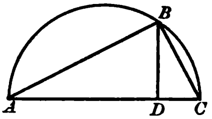


19. A and B are points on the opposite sides of a river. The point C is taken on a line with A and B, and CD and BE are made parallel. The line of sight from D to A passes through E. The measured values of CD, BE, and BC are 500 feet, 450 feet, and 120 feet respectively. Compute the length of AB.

20. The lengths of the sides of a triangle are 5 inches, 12 inches, and 13 inches respectively. What are the lengths of the lines connecting the middle points of its sides?

21. What is the perimeter of the triangle formed by connecting the mid-points of the sides of a triangle whose sides are 14 inches, 20 inches, and 25 inches respectively?

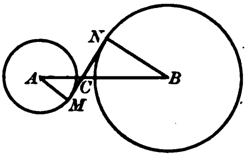
Fact from Geometry. — It is proved in geometry that if the angle ABC is a right angle and BD is drawn perpendicular to AC, then BD is a mean proportional between AD and DC.



22. What is BD, if $AC = 20$ and $AD = 16$? What is AC when $BD = 10$ and $AD = 15$?

23. By making use of the fact that $\overline{BD}^2 + \overline{DC}^2 = \overline{BC}^2$, prove that BC is a mean proportional between AC and DC.

24. In the accompanying figure MN is a tangent to each of two circles of radii 5 inches and 10 inches respectively, with centers 18 inches apart. Find the length of MN.



Suggestion. — From the fact that the triangles AMC and BNC are similar, find AC and BC. Then find MC and NC by making use of the fact that these triangles are right triangles.

25. Prove that if $\frac{a}{b} = \frac{c}{d}$, then $\frac{a}{c} = \frac{b}{d}$, and $\frac{b}{a} = \frac{d}{c}$.

These two relations are said to have been obtained from the given one by *alternation* and *inversion* respectively.

26. Prove that if $\frac{a}{b}=\frac{c}{d}$ then $\frac{a+b}{b}=\frac{c+d}{d}$.

Hint. — Add 1 to each member of the given equation and simplify each member of the resulting equation.

27. Prove that if $\frac{a}{b}=\frac{c}{d}$ then $\frac{a-b}{b}=\frac{c-d}{d}$.

28. Prove that if $\frac{a}{b}=\frac{c}{d}$ then $\frac{a+b}{a-b}=\frac{c+d}{c-d}$.

The resulting proportions in Exercises 26–28 are said to have been derived from the given proportion by *composition,* by *division,* and by *composition* and *division* respectively.

29. Prove that if $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$ then $\frac{a+c+e}{b+d+f}=\frac{a}{b}$.

Hint. — Let r be the value of each of the given fractions; that is, let $\frac{a}{b}=r$, $\frac{c}{d}=r$, $\frac{e}{f}=r$. Then clear each of these equations of fractions and equate the sum of the left members of the resulting equations to the sum of the right members. Find the value of r from this last equation.

A similar result holds for any number of equal ratios, and may be stated as follows: *In a series of equal ratios the sum of the antecedents is to the sum of the consequents as any antecedent is to its consequent.*

CHAPTER XV

QUADRATIC EQUATIONS

149. In § 108 the student learned how to solve certain quadratic equations, or quadratics, as they are sometimes called. The method described there is theoretically applicable to every quadratic, but in practice is subject to some limitation because of our inability to recognize the factors of every trinomial.

For example, we should have difficulty in attempting to solve the equation

$$x^2 + 4x + 1 = 0$$

by this method. In such cases we can proceed as follows:

$$x^2 + 4x + 1 = 0. \qquad (1)$$

Transpose 1: $$x^2 + 4x = -1. \qquad (2)$$

Add 4 to each member of (2):

$$x^2 + 4x + 4 = 3. \qquad (3)$$

Extract the square root of each member of (3):

$$x + 2 = +\sqrt{3}, \qquad (4)$$

or, $$x + 2 = -\sqrt{3}. \qquad (5)$$

Hence, $$x = -2 + \sqrt{3},$$

or, $$x = -2 - \sqrt{3}.$$

Check.
$$\begin{aligned}&(-2+\sqrt{3})^2 + 4(-2+\sqrt{3}) + 1\\ &= 4 - 4\sqrt{3} + 3 - 8 + 4\sqrt{3} + 1\\ &= 4 + 3 - 8 + 1 - 4\sqrt{3} + 4\sqrt{3}\\ &= 0;\end{aligned}$$

and
$$\begin{aligned}&(-2-\sqrt{3})^2 + 4(-2-\sqrt{3}) + 1\\ &= 4 + 4\sqrt{3} + 3 - 8 - 4\sqrt{3} + 1\\ &= 4 + 3 - 8 + 1 + 4\sqrt{3} - 4\sqrt{3}\\ &= 0.\end{aligned}$$

This work from equation (4) on can be condensed in the following way:

$$x + 2 = \pm\sqrt{3} \text{ (read "plus or minus } \sqrt{3}\text{").}$$

Hence, $$x = -2 \pm \sqrt{3} \text{ (read "} -2 \text{ plus or minus } \sqrt{3}\text{").}$$

Check.
$$\begin{aligned}&(-2 \pm \sqrt{3})^2 + 4(-2 \pm \sqrt{3}) + 1\\ &= 4 \mp 4\sqrt{3} + 3 - 8 \pm 4\sqrt{3} + 1\\ &= 4 + 3 - 8 + 1 \mp 4\sqrt{3} \pm 4\sqrt{3}\\ &= 0.\end{aligned}$$

In simplifying such condensed expressions the student should keep in mind that the upper signs of the double signs are to be used throughout, or else the lower signs throughout.

For example, in the work just given it would be incorrect to say that

$$\begin{aligned}&4 + 3 - 8 + 1 \mp 4\sqrt{3} \pm 4\sqrt{3}\\ &= 4 + 3 - 8 + 1 - 4\sqrt{3} - 4\sqrt{3},\end{aligned}$$

or, $$4 + 3 - 8 + 1 + 4\sqrt{3} + 4\sqrt{3}.$$

Returning now to the original equation, the student should verify that $$x^2 + 4x + 1 = (x + 2 - \sqrt{3})(x + 2 + \sqrt{3}).$$

And, having satisfied himself of this, he will see that we could have solved the equation by the method of factoring if we could have seen what these factors were.

150. The method that we have used here for solving a quadratic is called the **method of completing the square**, because the changing of the left member of the equation into a perfect square is an essential feature of it. It is applicable to the solution of every quadratic, and rests upon the following considerations:

When a trinomial square is arranged according to the descending powers of one of its letters, the second term of its square root is equal to the quotient obtained by dividing the second term of the trinomial by twice the square root of its first term.

Thus a square root of $a^2 + 2ab + b^2$ is $a + b$ and $b = \frac{2ab}{2a}$. (See § 131.)

Hence, if the last term of the trinomial square is missing, the trinomial can be completed by adding to the first two terms the square of this quotient.

In applying this method we should have the terms involving the unknown on the left side of the equation and the known terms on the right side. Then the term to be added to each side to complete the square will not contain the unknown.

In solving a quadratic, never complete the square by adding to each side a term that contains the unknown.

Solve $$x^2 + 5x + 2 = 0. \tag{1}$$

Transposing, $$x^2 + 5x = -2. \tag{2}$$

Completing the square, $$x^2 + 5x + (\tfrac{5}{2})^2 = -2 + (\tfrac{5}{2})^2, \tag{3}$$

or $$x^2 + 5x + \tfrac{25}{4} = \tfrac{17}{4}.$$

Extracting the square root of each member,

$$x + \frac{5}{2} = \pm\frac{\sqrt{17}}{2}. \tag{4}$$

$$x = -\frac{5}{2} \pm \frac{\sqrt{17}}{2} = \frac{-5 \pm \sqrt{17}}{2}. \tag{5}$$

151. Any number that answers the description given by equation (1) also answers the descriptions given by (2) and (3) respectively. Moreover, if two numbers are equal, they must have the same square roots, and if we select one of the roots of the first number, we know that it must be equal to one or the other of the roots of the second number. Hence any number that answers the description given by (3) must answer one of the two descriptions given by (4), and any number that answers either of these descriptions must answer the description given by (3). By these successive steps, then, we reduce the problem of solving a quadratic to the problem of solving two linear equations.

152. When the coefficient of x^2 is not a perfect square, it is best to multiply or to divide both members of the equation by a number so selected that in the resulting equation the coefficient of x^2 is a perfect square.

Solve
$$5x^2 + 3x - 1 = 0.$$
$$x^2 + \tfrac{3}{5}x = \tfrac{1}{5}.$$
$$x^2 + \tfrac{3}{5}x + \tfrac{9}{100} = \tfrac{1}{5} + \tfrac{9}{100} = \tfrac{29}{100}.$$
$$x + \frac{3}{10} = \pm\frac{\sqrt{29}}{10}.$$
$$x = \frac{-3 \pm \sqrt{29}}{10}.$$

We can also solve the equation in the following way:
Multiply both members of the equation by 20.

$$100x^2 + 60x - 20 = 0.$$
$$100x^2 + 60x = 20.$$
$$100x^2 + 60x + 9 = 29.$$
$$10x + 3 = \pm\sqrt{29}.$$
$$x = \frac{-3 \pm \sqrt{29}}{10}.$$

The advantage of this second method is that it avoids fractions until the very end. This result can always be accomplished by multiplying both members of the equation by four times the coefficient of x^2. This is the *Hindu Method* for solving a quadratic.

Historical Note. — The Hindu Method of solving a quadratic equation was invented by Bhâskara, who was born in 1114 A.D.

153. Definition. — A quadratic equation that does not contain the first power of the unknown is called a **pure quadratic.**

Thus $x^2 - 5 = 0$ is a pure quadratic.
In solving a pure quadratic it is unnecessary to complete the square.

EXERCISES

Solve the following quadratic equations, using the method of factoring whenever you can. Do not compute the approximate values of the irrational roots.

1. $x^2 - 6x + 7 = 0.$
2. $y^2 = 2y + 1.$
3. $x^2 + 3x = 0.$
4. $z^2 + 12z + 15 = 0.$
5. $x^2 = 10 - 6x.$
6. $3m^2 - 9m + 2 = 0.$

7. $y^2 + 13y + 14 = 0.$

8. $x^2 - 11x + 30 = 0.$

9. $5y^2 - 13 = 0.$

10. $x^2 - 2\frac{1}{2}x - 3 = 0.$

11. $4 = 2x + 3x^2.$

12. $5y^2 - 30 = 0.$

13. $(x-2)(2x+1) = 5 - 3x.$

14. $\frac{x^2}{3} + \frac{5x}{2} = 1 - 3x.$

15. $3y + 9 = 2y^2.$

16. $6x = 1 + 9x^2.$

17. $12x^2 = 7 - 4x.$

18. $2y + 1 = 3y^2.$

19. $1 - 5x^2 = 2x.$

20. $15y^2 - 4y = 1.$

21. $\frac{x}{13} = \frac{12}{x}.$

22. $\frac{2}{R+1} = \frac{5(R-1)}{7}.$

23. $x^2 + 2x + 5 = 9.$

24. $M^2 - 3M - 2 = 0.$

25. $\frac{x-1}{x+1} - \frac{x+1}{x-1} = 5.$

26. $\frac{3k}{4} + \frac{2}{3} = \frac{1}{2}k^2.$

27. $\frac{a+5}{6} - \frac{4}{a+1} = 2.$

28. $\frac{x^2-x+1}{x+1} + 3x = \frac{x^2+x+1}{x+1}.$

29. $x + \frac{1}{x} = 2.$

30. $\frac{1}{y+1} + \frac{1}{y+2} + \frac{1}{y+3} = 0.$

31. $2(x+5)^2 - 17 = 3(x+5).$

32. $\frac{x+3}{x+9} = 3\frac{x+4}{x+1}.$

33. $(a+5)(2a-3) = 4 - 3a(a+1).$

34. $\frac{1}{r} + \frac{1}{r+1} = 5.$

35. $\frac{x-3}{x-5} = \frac{x-4}{x-7} - 4.$

36. $\frac{z}{z-1} + \frac{z+1}{z} = 1.$

37. $x^2 + 3x = 3(2+x) - 5.$

38. $\frac{2}{x+1} + \frac{2(x-1)}{x} + \frac{1}{x^2+x} = 0.$

39. Solve the equation $\frac{x^2}{4} - \frac{y^2}{9} = 1$ for x when $y = 2$.

40. Solve the equation $x^2 + 2xy + y^2 + 3x - 2y = 0$ for x when $y = 1$.

Find the values of the roots of the following equations correct to two decimal places:

41. $x^2 + x - 1 = 0.$
42. $2x^2 + 2x - 1 = 0.$
43. $3x^2 - 4x - 5 = 0.$
44. $2x^2 + 4x - 21 = 0.$
45. $6x + 1 = 3x^2.$
46. $2x^2 - x = 20.$
47. $2x^2 + 3 = -6x.$
48. $3x^2 + 4x = 7.$
49. $5 + x = x^2.$
50. $2x^2 + 8x + 3 = 0.$

154. Quadratics with Literal Coefficients. — The same methods can be used for the solution of quadratics some of whose coefficients are literal.

Solve: $x^2 + (a+1)x + 2a = 0.$

$$x^2 + (a+1)x = -2a.$$

$$x^2 + (a+1)x + \frac{(a+1)^2}{4} = -2a + \frac{(a+1)^2}{4} = \frac{a^2 - 6a + 1}{4}.$$

$$x + \frac{a+1}{2} = \pm\frac{\sqrt{a^2 - 6a + 1}}{2}.$$

$$x = \frac{-a - 1 \pm \sqrt{a^2 - 6a + 1}}{2}.$$

EXERCISES

Solve the following quadratics:

1. $y^2 + 4by = 0.$
2. $(ax + b)^2 = 16.$
3. $x^2 + (ax + 1)^2 = 1.$
4. $x^2 + a^2(x + 1)^2 = a^2.$
5. $y^2 + (6y + l)^2 = l^2.$
6. $3x^2 + 4x + r = 0.$
7. $5ax = 6 + a^2x^2.$
8. $2x(x + c) + c^2 - 2 = 0.$
9. $x^2 + (x - c\sqrt{2})^2 = 2c^2.$
10. $x^2 + (x + c\sqrt{2})^2 = c^2.$
11. $x^2 + 3x + m = 0.$
12. $x^2 + (1 + m)x + m = 0.$
13. $(3x - 2)^2 = 4px.$
14. $y^2 - 6y = 1 + m.$

15. $r^2 + 8ar + 15a^2 = 0.$

16. $x^2 + 2x + (1 - k^2) = 0.$

17. $6m^2x^2 + 5mx = 2.$

18. $y^2 + (3y - 2)^2 = r^2.$

19. $y^2 - (3y - 2)^2 = r^2.$

20. $(1 + k)x^2 + 2kx = 2.$

21. $4x^2 + (1 + k)x + 1 = 0.$

22. $x^2 + (mx + 1)^2 = 9.$

23. $x^2 + bx + c = 0.$

24. $ax^2 + x + c = 0.$

25. $ax^2 + bx + 1 = 0.$

26. $4y^2 + 9(2y + a) = 36.$

27. $x^2 - 8x + 12\frac{3x - c}{4} - 44 = 0.$

28. $y^2 - 5(my + 2) + 4y - 10 = 0.$

29. $3x^2 + 12x + 4\frac{x - b}{3} - 8 = 0.$

30. $5y - 1 = 3y^2 + 5\frac{y - c}{3}.$

31. $ax^2 + bx + c = 0.$

155. Solution by Formulæ. — From the solution of Exercise 31 of § 154 we find that

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

are the roots of the equation $ax^2 + bx + c = 0$.

Now by giving suitable values to a, b, and c in this equation we can get any quadratic. Hence this equation can be considered the *typical*, or *general*, quadratic, and by substituting the proper values for a, b, and c in the expressions for x_1 and x_2 we can get the roots of any quadratic. This is the usual way of solving a quadratic, and the student should accordingly commit to memory the formulæ for x_1 and x_2 and use them hereafter whenever he has to solve such an equation, unless he can get the solution more readily by factoring.

He should be careful to have the equation in the standard form given above before using the formulæ.

EXERCISES

Solve the following equations by using the formulæ, and check your results:

1. $x^2+x-1=0.$
2. $3x^2+x-1=0.$
3. $2y+1=y^2-5.$
4. $8x^2+5x=5.$
5. $x^2+4x+3=0.$
6. $x^2-x-1=0.$
7. $3x^2+4x+1=0.$
8. $x^2-10=0.$
9. $x^2=6x+10.$
10. $9m=3m^2+4.$
11. $\frac{x+1}{3}=\frac{2}{5(x-1)}.$
12. $\frac{1}{x-1}+\frac{1}{x-2}+\frac{1}{x-3}=0.$
13. $\frac{1}{r}+\frac{1}{r+1}=3.$
14. $2\frac{x-1}{x+2}=\frac{x+3}{x+4}.$
15. $14x^2=3x.$
16. $\frac{r-2}{4}-\frac{5}{r+1}=2.$
17. $x+\frac{1}{x}=6.$
18. $y=\frac{2}{y}.$
19. $\frac{9}{y}=\frac{y}{5}.$
20. $(3x-1)^2=4x.$
21. $-30+x+x^2=0.$
22. $y^2+5y+1=0.$
23. $(x+1)(x+3)=x+5.$
24. $\frac{2}{x}+\frac{x}{5}=3.$
25. $x^2+10x=0.$
26. $\frac{x^2}{4}+\frac{(2x-1)^2}{9}=1.$
27. $y^2=4m(3y+1).$
28. $x^2+(x+1)^2=r^2.$
29. $\frac{x+5}{x-2}=2\frac{x+1}{x+4}.$
30. $15m^2+4m=5.$
31. $3x^2+x+2=6x.$
32. $7x^2=0.$
33. $\frac{(2y-6)^2}{9}-\frac{y^2}{4}=1.$
34. $\frac{1}{x-1}+x-1=9.$
35. $\frac{1}{x}+\frac{x}{2}=10.$
36. $(5x-11)^2=6(x-2).$
37. $(3y-1)^2+\frac{y^2}{4}=1.$

38. Solve the equation $\frac{x^2}{4}+\frac{y^2}{9}=1$ for x when $y=1$.

39. Solve the equation $x^2+y^2+3x+y+2=0$ for x when $y=-1$.

40. Solve the equation $(y+2)^2=8(x-3)$ for y when $x=4$.

156. If we should attempt to solve the equation

$$x^2+2x+2=0$$

by completing the square, we should get

$$x^2+2x+1=-1.$$

In this new equation the left member is a perfect square and the right member is -1. But it follows from the Rule of Signs for multiplication that a negative number cannot be the square of either a positive or a negative number. We conclude, therefore, that there is no number that answers the description given by this equation. But in the Second Course we shall see that there is a kind of number which the student has not yet studied whose square is a negative number.

157. Equations involving Radicals. — Certain equations involving radicals can be solved by the methods described in the following illustrative examples.

It is understood that in these examples we are to attach to every radical its positive value. Thus $\sqrt{9}$ represents 3, and not -3. The latter is represented by the symbol $-\sqrt{9}$.

Example 1. Solve the equation

$$\sqrt{x^2-9}-4=0.$$

Transposing -4 to the right member, we get

$$\sqrt{x^2-9}=4.$$

Squaring both members, $x^2-9=16,$

$$x^2=25,$$

$$x=\pm 5.$$

Check. $\sqrt{25-9}-4=4-4=0.$

[illegible] polynomials [illegible] the sake [illegible] equation with [illegible] in reality [illegible] our work. Hence [illegible] the solution [illegible] next example.

[illegible]

[illegible] As a matter of [illegible]

[illegible]

[illegible] member.

[illegible]

$[illegible] = 29 - 21 = 5.$

[illegible]

$[illegible] = 5.$

[illegible] member.

[illegible]

$[illegible] - 41 - x^2 + 10x + 41,$

[illegible]

Squaring both members,

$$25x^2 + 160x + 256 = 16x^2 + 160x + 656,$$
$$9x^2 = 400,$$
$$x = \pm\frac{20}{3}.$$

Check for $\frac{20}{3}$.

$$\sqrt{\frac{400}{9} - \frac{200}{3} + 41} - \sqrt{\frac{400}{9} + \frac{200}{3} + 41} = \sqrt{\frac{169}{9}} - \sqrt{\frac{1369}{9}} = \frac{13}{3} - \frac{37}{3} = -8.$$

Hence $\frac{20}{3}$ is not a root of the equation.

Check for $-\frac{20}{3}$.

$$\sqrt{\frac{400}{9} + \frac{200}{3} + 41} - \sqrt{\frac{400}{9} - \frac{200}{3} + 41} = \sqrt{\frac{1369}{9}} - \sqrt{\frac{169}{9}} = \frac{37}{3} - \frac{13}{3} = 8.$$

Hence $-\frac{20}{3}$ is a root of the equation.

158. In solving equations of this kind the student should proceed in accordance with the

Rule. — *If necessary, so transpose all the terms that one radical stands alone in one member of the equation.*

Then form a new equation by equating the squares of the members of the last equation.

If the resulting equation contains radicals, repeat the process until an equation is obtained that is free from radicals.

Solve this last resulting equation and test each of its roots in the original equation.

In simplifying the radicals in order to determine whether a number satisfies the equation, the student must not resort to squaring both members of the equation, since the question he is trying to settle is whether every root of the equation formed in this way is a root of the original equation.

Certain equations involving radicals can be solved best by the method explained in the next article.

EXERCISES

Solve the following equations and check your results:

1. $\sqrt{x+2}-x=0$.

2. $\sqrt{2x+5}=\sqrt{x+10}$.

3. $\sqrt{4x+13}+x=2$.

4. $\dfrac{\sqrt{x+9}}{3}+\dfrac{1}{4}=2$.

5. $\sqrt{x^2+3x+2}=x+1$.

6. $\dfrac{\sqrt{x+16}}{4}+\dfrac{x+9}{5}=\dfrac{97}{20}$.

7. $\sqrt{x^2+10x+41}-\sqrt{x^2-10x+41}=8$.

8. $\sqrt{x^2-8x+24}+\sqrt{x^2+8x+24}=10$.

9. $x+\sqrt{x(6+x)}=\frac{9}{2}$.

10. $\sqrt{(x+2)^2+3}+\sqrt{(x-1)^2+3}=5$.

11. $\sqrt{10x-3}=\sqrt{x^2+6x+1}$.

12. $\sqrt{9x+1}=\sqrt{x^2+3x+9}$.

13. $\sqrt{5x-6}-\sqrt{x-2}=2$.

14. $\sqrt{3x+10}=2\sqrt{3x+1}-\sqrt{x+4}$.

15. $1125=1093\sqrt{1+.003665\,t}$.

16. $v=1093\sqrt{1+.003665\,t}$.

Solve for t.

This formula gives the velocity of sound in air in feet per second at a temperature of $t°$ Centigrade. The formula of Ex. 15 is a particular case of this.

159. Equations in the Form of Quadratics. — We can frequently apply the methods for solving quadratics to the solution of equations that involve the unknown only in a certain expression and in the square of this expression.

Example 1. Solve the equation

$$3x^4 - 14x^2 + 8 = 0.$$

This can be written

$$(x^2 - 4)(3x^2 - 2) = 0.$$

Hence $\qquad x^2 = 4 \text{ or } \tfrac{2}{3},$

and $\qquad x = \pm 2 \text{ or } \pm\sqrt{\tfrac{2}{3}}.$

The student should check these results.

Example 2. Solve the equation

$$3x - 5 - 4\sqrt{3x - 5} + 3 = 0.$$

This can be written

$$(\sqrt{3x - 5})^2 - 4\sqrt{3x - 5} + 3 = 0.$$

Hence $\qquad \sqrt{3x - 5} = 3 \text{ or } 1,$

and $\qquad 3x - 5 = 9 \text{ or } 1,$

$$x = \tfrac{14}{3} \text{ or } 2.$$

The student should check these results.

Example 3. Solve the equation

$$x^2 - 3x + 5 - 7\sqrt{x^2 - 3x + 8} = -15.$$

This can be changed into a quadratic in $\sqrt{x^2 - 3x + 8}$ by adding 3 to each member. Thus,

$$x^2 - 3x + 8 - 7\sqrt{x^2 - 3x + 8} = -12.$$

Hence $\qquad \sqrt{x^2 - 3x + 8} = 3 \text{ or } 4,$

and $\qquad x^2 - 3x + 8 = 9 \text{ or } 16,$

$$x = \frac{3 \pm \sqrt{13}}{2} \text{ or } \frac{3 \pm \sqrt{41}}{2}.$$

The student should check these results.

EXERCISES

Solve the following equations and check your results:

1. $x^4 - 5x^2 + 4 = 0.$
2. $x^4 - 5x^2 + 6 = 0.$
3. $x^4 - 9x^2 + 20 = 0.$
4. $2x^4 - 11x^2 + 5 = 0.$
5. $x^4 - 2x^2 + 1 = 0.$
6. $y^4 - 10y^2 + 16 = 0.$
7. $\left(x + \frac{3}{x}\right)^2 + \left(x + \frac{3}{x}\right) = 20.$
8. $(y^2 - 2)^2 - 2(y^2 - 2) - 3 = 0.$

9. $(x^2+4x-1)^2+x^2+4x=3.$

10. $(2x^2-3x+1)^2-10x^2+15x-1=0.$

11. $x^2+x-6\sqrt{x^2+x}+5=0.$

12. $(x^2-2)^2+3(x^2-2)+2=0.$

13. $x^2-4x-7\sqrt{x^2-4x+5}+15=0.$

14. $2\sqrt{x^2+x-6}+x^2+x=21.$

15. $\left(x+\frac{1}{x}\right)^2+2\left(x+\frac{1}{x}\right)-8=0.$

PROBLEMS

1. The product of two consecutive integers is 650. What are they?

2. Find two consecutive odd integers whose product is 783.

3. Find two consecutive integers the sum of whose squares is 545.

4. Two automobiles are 75 miles apart on roads at right angles to each other. They meet at the crossing after traveling 2 hours. How fast did they travel, supposing that one of them went 50 per cent faster than the other one?

5. An open tin box with a square base and 4 inches deep is to be made by cutting out square corners from a piece of tin and turning up the sides. How large a piece of tin must be used in order that the box shall contain 484 cubic inches?

6. What must be the dimensions of a rectangular piece of tin three times as long as it is wide from which can be made an open box 5 inches deep such that the area of the sides and ends is twice the area of the bottom?

7. A lever is to be cut from a bar weighing 4 pounds to the foot. How long must it be in order that it may balance about a point 3 feet from one end when a weight of 40 pounds is attached to this end? (See § 73.)

8. How long a lever weighing 2 pounds to the foot will balance about a point 4 feet from one end when a weight of 15 pounds is attached to this end and one of 10 pounds is attached to the other end?

9. A lever turning freely about an axis at one end is acted on by two parallel forces; namely, —

An upward force of 15 pounds at $1\frac{1}{2}$ feet from the axis.

A downward force of 5 pounds at the other end.

If the lever weighs 3 pounds to the foot, how long must it be that it may be in equilibrium?

10. An object falls s feet in t seconds, where

$$s = 16\ t^2.$$

How long will it take to fall 80 feet?

11. If an object is thrown vertically upwards with a velocity of v feet a second, in t seconds it will be s feet from the starting point, where

$$s = vt - 16\ t^2.$$

When will an object thrown vertically upwards with a velocity of 50 feet a second be 20 feet above the starting point? How do you account for the two answers?

12. When will the object referred to in the preceding problem be 39 feet above the starting point?

13. When an object is thrown vertically downwards with a velocity of v feet per second, the number s of feet fallen in t seconds is given by the formula

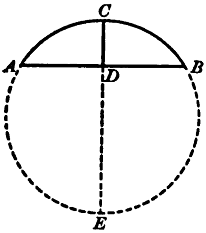
$$s = vt + 16\ t^2.$$

An object is thrown downwards with a velocity of 15 feet a second from a point 135 feet above the earth. How long will it be until it strikes the ground?

14. When the equation

$$s = vt + 16\,t^2$$

is solved for t, two solutions are obtained. How do you account for this? (See § 110.)



In a circular arch ACB we call the distance AB the *span*, the distance DC from the middle of AB to the highest point of the arch the *rise*, and the diameter of the completed circle the *diameter* of the arch.

It is shown in geometry that DB is a mean proportional between CD and DE.

15. Find the span of an arch whose diameter is 18 feet and rise 2 feet.

16. Find the span of an arch whose rise is a feet and diameter b feet.

17. A man wishes to lay out a rectangular field to contain 2 acres and have a perimeter of 72 rods. How long should it be?

18. How long must a rectangular field containing 10 acres be in order that its perimeter shall be 160 rods?

19. What are the dimensions of a rectangle whose area is b square units and whose perimeter is p linear units?

20. The owner of a rectangular piece of land 100 yards by 60 yards wishes to enlarge it into a rectangular piece containing twice as much land by adding strips of equal width to one end and one side. How wide should these strips be?

21. A certain rectangle is 10 feet long and 6 feet wide. What must be the dimensions of a second rectangle twice as large whose sides are proportional to those of the first rectangle?

22. The dimensions of a rectangle are a units and b units respectively. A second rectangle is k times as large as this one, and the dimensions of the two are proportional. What are the dimensions of the larger rectangle?

23. A certain rectangle is x units longer and x units wider than another rectangle a units long and b units wide. The area of the larger rectangle is r times the area of the smaller one. Find the length of the larger rectangle in terms of a, b, and r.

24. The abscissa of the point A is 5 and that of B is 20. Find the abscissa of a point P such that AP is a mean proportional between AB and PB.

25. A and B are two points with the abscissæ 3 and 5 respectively. What must be the abscissa of X in order that AX shall be a mean proportional between AB and XB?

26. If -8 and 12 are the abscissæ of A and B respectively, what must be the abscissa of X in order that BX shall be a mean proportional between BA and XA?

27. How much must be added to the length of a rectangle 8 inches by 6 inches in order to increase the diagonal by 2 inches?

28. A rectangle is 16 feet by 12 feet. Another rectangle has the same perimeter and a diagonal of 21 feet. What are the dimensions of this second rectangle?

29. A man in a boat is 3 miles from the nearest point of the beach, which is straight, and he wishes to reach a point on the beach 4 miles from this nearest point in $1\frac{3}{4}$ hours. He can row 3 miles an hour and walk 4 miles an hour. At what point should he land?

Let $x =$ distance in miles of his landing place from the nearest point on the beach.

Then $\sqrt{x^2+9} =$ number of miles to be rowed.

$4 - x =$ number of miles to be walked.

$\frac{\sqrt{x^2+9}}{3} =$ number of hours taken for rowing.

$\frac{4-x}{4} =$ number of hours taken for walking.

$$\frac{\sqrt{x^2+9}}{3} + \frac{4-x}{4} = 1\tfrac{3}{4}.$$

Transpose all the terms not involving the radical sign to the right side of the equation:

$$\frac{\sqrt{x^2+9}}{3}=\frac{3+x}{4}.$$

Square both members of this equation:

$$\frac{x^2+9}{9}=\frac{9+6x+x^2}{16}.$$

Solve for x.

30. A man in a boat 2 miles from the nearest point of a straight beach wishes to reach a point on the beach 6 miles from this nearest point in 2 hours. If he rows 3 miles an hour, and walks 4 miles an hour, at what point should he land?

CHAPTER XVI

SYSTEMS OF EQUATIONS SOLVABLE BY QUADRATICS

160. We can solve a linear equation and a quadratic equation in two variables as illustrated in the following example:

Solve
$$\begin{cases} 3x - y + 2 = 0, & (1) \\ x^2 + y^2 = 4. & (2) \end{cases}$$

Solving (1) for y, we get
$$y = 3x + 2. \tag{3}$$

Substituting this value of y for y in (2), we get
$$x^2 + (3x + 2)^2 = 4, \tag{4}$$
$$10x^2 + 12x = 0, \tag{5}$$
$$2x(5x + 6) = 0,$$
$$x = 0, \text{ or } -\tfrac{6}{5}.$$

Substituting these values of x for x in (3), we get
$$y = 2, \text{ or } -\tfrac{8}{5}.$$

Hence the original equations are satisfied by the pairs of values
$$x = 0, \quad x = -\tfrac{6}{5}.$$
$$y = 2, \quad y = -\tfrac{8}{5}.$$

The student should check these results.

If we had substituted the values of x for x in equation (2) we should have got $y = \pm 2$ and $\pm \frac{8}{5}$ respectively. But equation (1) is not satisfied for $x = 0$, $y = -2$, nor for $x = -\frac{6}{5}$, $y = \frac{8}{5}$. In general, in problems of this kind, when the values of one variable have been found, these values should be substituted for this variable in the linear equation in order to get the corresponding values of the other variable.

EXERCISES

Solve the following systems of equations. In the case of irrational roots get the results correct to two decimal places. Indicate clearly what values of x and y are to be taken together.

1. $x - y = 1,$
$x^2 + y^2 = 5.$

2. $x + y = 6,$
$x^2 + y^2 = 20.$

3. $x + y = 2,$
$x^2 + y^2 = 20.$

4. $x + 2y = 1,$
$x^2 + y^2 = 1.$

5. $y = 2x + 1,$
$x^2 + y^2 = 10.$

6. $3x - y - 2 = 0,$
$x^2 + y^2 = 20.$

7. $3x - 4y - 2 = 0,$
$xy = 2.$

8. $y = x + \sqrt{2},$
$x^2 + y^2 = 1,$

9. $3x - 4y + 6 = 0,$
$x^2 + y^2 - 8x + 12y - 33 = 0.$

10. $x = 0,$
$x^2 + y^2 + 2x + 2y + 1 = 0.$

11. $y = 2x + 1,$
$x^2 + y^2 + 10x - 6y - 11 = 0.$

12. $2x + 3y + 1 = 1,$
$y^2 - 5x + 4y - 10 = 52.$

13. $2x + y = 1,$
$3x^2 + 12x + 4y = 8.$

14. $5x - 2y + 3 = 0,$
$xy = 1.$

15. $y = 2x - 1,$
$5x^2 + 5y^2 - 1 = 0.$

16. $x^2 - y^2 = 0,$
$2x + y = 2.$

17. $\frac{x^2}{4} - \frac{y^2}{9} = 1,$
$4y - x = 0.$

18. $\frac{x^2}{4} - \frac{y^2}{9} = 1,$
$y = x + 1.$

19. $\frac{x^2}{4} + \frac{y^2}{9} = 1,$
$x + y = 2.$

20. $y^2 = 4x,$
$x + y - 1 = 0.$

161. It may be that each equation contains each unknown only to the second power. Consider, for example, the equations

$$x^2 + y^2 = 25,$$
$$\frac{x^2}{36} + \frac{y^2}{9} = 1.$$

Eliminating x^2, we get
$$y^2 = \frac{33}{9},$$
$$y = \pm\frac{\sqrt{33}}{3}.$$

Hence
$$x = \pm\frac{8\sqrt{3}}{3}.$$

The student should verify that the equations are satisfied for either of these values of x in connection with either of the values of y.

Sometimes the equations are of the form
$$x^2 + y^2 = 16, \tag{1}$$
$$xy = 4\tfrac{1}{2}. \tag{2}$$

These equations can be solved as follows:

Multiply each member of (2) by 2:
$$2\,xy = 9. \tag{3}$$

Equate the sum of the left members of (1) and (3) to the sum of the right members:
$$x^2 + 2\,xy + y^2 = 25. \tag{4}$$

Extract the square root of each member of (4):
$$x + y = \pm 5. \tag{5}$$

Hence
$$y = 5 - x \tag{6}$$

or
$$y = -5 - x. \tag{7}$$

From (2) and (6) we get
$$x(5 - x) = 4\tfrac{1}{2},$$
$$2\,x^2 - 10\,x + 9 = 0,$$
$$x = \frac{5 \pm \sqrt{7}}{2}.$$

From (6)
$$y = 5 - \frac{5 \pm \sqrt{7}}{2} = \frac{5 \mp \sqrt{7}}{2}.$$

In a similar way we get from (2) and (7)
$$x = \frac{-5 \pm \sqrt{7}}{2},$$
$$y = \frac{-5 \mp \sqrt{7}}{2}.$$

Hence equations (1) and (2) are satisfied by the following pairs of values of x and y.

$$x = \frac{5+\sqrt{7}}{2},\ \frac{5-\sqrt{7}}{2},\ \frac{-5+\sqrt{7}}{2},\ \frac{-5-\sqrt{7}}{2}.$$

$$y = \frac{5-\sqrt{7}}{2},\ \frac{5+\sqrt{7}}{2},\ \frac{-5-\sqrt{7}}{2},\ \frac{-5+\sqrt{7}}{2}.$$

Any of these values of x together with any of the values of y except the one immediately below it do not satisfy these equations. In the preceding example, on the other hand, we found that either of the values of x and either of the values of y satisfied the equations.

162. A set of equations of the form

$$x^3 + y^3 = 8, \qquad (1)$$

$$x + y = 2, \qquad (2)$$

can be solved as follows:

Write (1) in the form

$$(x+y)(x^2 - xy + y^2) = 8,$$

and substitute for $x + y$ its value, as given by (2):

$$x^2 - xy + y^2 = 4. \qquad (3)$$

Equations (2) and (3) can be solved in the way explained in § 160.

EXERCISES

Solve the following equations and indicate how the values of x and y are to be associated:

1. $x^2 + y^2 = 13,\quad \frac{x^2}{2} - \frac{y^2}{9} = 1.$

2. $x^2 + y^2 = 16,\quad \frac{x^2}{4} - \frac{y^2}{9} = 4.$

3. $x^2 + y^2 = 9,\quad \frac{x^2}{4} - \frac{y^2}{9} + 1 = 0.$

4. $x^2 + y^2 = 3,\quad xy = \frac{1}{2}.$

5. $x^2 + y^2 = 14,\quad xy = 1.$

6. $x^2 + y^2 = 7,\quad xy = 1.$

7. $x^3 - y^3 = 125,\quad x - y = 5.$

8. $x^3 + 8y^3 = 2,\quad x + 2y = 2.$

9. $y^3 - 2 = x^3,\quad y = 2 + x.$

10. $x^2 + y^2 = 9,\quad x^2 + 4y^2 = 12.$

11. $xy = 1,\quad x^2 + y^2 = 38.$

12. $x^2+\frac{y^2}{5}=1,$
$x^2+y^2=4.$

13. $\frac{x^2}{9}+\frac{y^2}{4}=1,$
$3\,xy=7.$

14. $\frac{x^2}{5}+\frac{y^2}{6}=1,$
$\frac{x^2}{4}-\frac{y^2}{3}=1.$

15. $27x^3+8y^3=2,$
$3\,x+2\,y=2.$

16. $x^2+y^2=9,$
$\frac{x^2}{2}+\frac{y^2}{16}=1.$

17. $x^2+y^2=16,$
$\frac{x^2}{2}-\frac{y^2}{3}=1.$

18. $x^4+x^2y^2+y^4=4,$
$x^2-xy+y^2=2.$

19. $x^4-x^2y^2+16\,y^4=9,$
$x^2+3\,xy+4\,y^2=3.$

20. $x^4-y^4=12,$
$x^2-y^2=3.$

In Exs. 18–20 use the general method of § 162.

PROBLEMS

1. The sum of the squares of two numbers is 146 and the product of the numbers is 55. What are the numbers?

2. The sum of the squares of two numbers is 65 and the sum of the numbers is 11. What are the numbers?

3. The sum of the lengths of two cubes is 8 inches and the sum of the volumes of the cubes is 224 cubic inches. What are the dimensions of each cube?

4. How long must the legs of a right triangle be in order that the hypotenuse be 10 inches and the area be 24 square inches?

5. What must be the dimensions of a rectangular field that contains 1 acre and has a diagonal of 18 rods?

6. The sum of the radii of two circles is 8 inches and the sum of their areas equals two thirds the area of a circle of radius 8 inches. What are the radii of the two circles?

7. What must be the dimensions of a rectangular field that contains 4 acres and can be inclosed by a fence 120 rods long?

8. What must be the dimensions of a rectangular field that contains a square rods and can be inclosed by a fence f rods long?

9. What is the area of a right triangle whose hypotenuse is 15 inches and perimeter 36 inches?

Hint. — First find the lengths of the sides.

10. What is the area of a right triangle whose hypotenuse is 13 inches and perimeter 30 inches?

11. What is the hypotenuse of a right triangle one of whose sides is 24 inches and whose perimeter is 60 inches?

12. What must be the dimensions of a rectangular card containing 30 square centimeters in order that the longest straight line which can be drawn on it shall be 13 centimeters?

13. What must be the dimensions of a rectangular card in order that the diagonal be 6 inches and the perimeter 14 inches?

14. Find the lengths of the sides of a right triangle whose perimeter is 24 inches and whose area is 24 square inches.

Hint. — Let $x =$ the length in inches of one of the legs of the triangle,
and $y =$ the length of the other leg.

Then the length of the hypotenuse is $\sqrt{x^2 + y^2}$.

Hence $x + y + \sqrt{x^2 + y^2} = 24$
and $xy = 48$.

Free the first equation of radicals and combine the resulting equation with the second equation.

15. What must be the lengths of the sides of a right triangle whose perimeter is 48 inches and whose area is 96 square inches?

16. A rectangular field is to be laid out whose perimeter shall be 68 rods and whose diagonal shall be 26 rods. What must be its dimensions?

17. Two cubical boxes differ in volume by 218 cubic inches and an edge of the smaller one is 2 inches shorter than an edge of the other one. What is the edge of each?

REVIEW OF CHAPTERS XIII–XVI

1. What is meant by saying that a polynomial is a perfect square?

2. What is a trinomial square?

3. What is meant by a square root of a perfect square?

4. Give the rule for finding a square root of a perfect square.

5. How many square roots has every perfect square?

6. Can an integer whose right-hand digit is 3 be a perfect square?

7. Name the digits that can be the right-hand digits of perfect squares.

8. Explain why the rule for finding a square root of a polynomial cannot be used without modification for finding square roots of numbers.

9. What is a rational number?

10. Is there a number that is exactly equal to the square root of 5?

11. Define a surd.

12. Is there any difference between an irrational number and a surd?

13. What is a radical?

14. When is a radical of index 2 said to be simplified?

15. How can you recognize a trinomial square?

16. Explain the rule for determining the number of digits in the square root of a number.

17. How can you find the fourth root of a number?

18. Find the length of a side of a square that is equal in area to a circle of radius 3 inches.

19. Find the radius of a circle that is equal in area to a square each side of which is 4 inches.

20. What is a ratio?

21. What is a proportion?

22. What is the most important property of a proportion that you can think of?

23. What is the third proportional to two numbers?

24. What is a mean proportional between two numbers?

25. How many mean proportionals are there between two numbers?

26. How can you find the mean proportionals between two numbers?

27. What is the fourth proportional to three numbers?

28. What is meant by saying that a, b, c, and d are proportional?

29. What is meant by the statement that if four numbers are in proportion they are in proportion by alternation?

30. What is meant by the statement that if four numbers are in proportion they are in proportion by inversion?

31. What is the effect on the value of a fraction of adding the same positive number to the numerator and the denominator, if the numerator and the denominator are both positive? (See Exercises 17, 18, p. 230.)

32. The sides of a right triangle are 6 inches, 8 inches, and 10 inches respectively. Find the length of the perpendicular from the vertex of the right angle to the hypotenuse. (See Exercise 22, p. 237.)

33. Find the distance from the middle of the hypotenuse of the triangle of Exercise 32 to the vertex of the right angle.

34. What is a quadratic equation?

35. Describe three ways of solving a quadratic equation.

36. If you could solve a quadratic by any one of these ways, which one would you prefer? Give a reason for your answer.

37. Describe in detail the process of solving a quadratic by completing the square.

38. How many numbers answer the description given by a quadratic equation?

39. What is a pure quadratic?

40. What is the Hindu Method of solving a quadratic, and what advantage is to be gained by its use?

41. Give the formulæ for the roots of the equation $ax^2 + bx + c = 0$.

42. Solve the equation $\sqrt{x^2 + 3x + 8} - \sqrt{x^2 + 3x - 3} = 1$.

43. Solve the equation $(x^2 + x - 2)^2 - 4(x^2 + x - 2) + 3 = 0$.

44. A man in a boat 3 miles from the nearest point of a straight beach wishes to reach a point on the beach 5 miles from this nearest point in 2 hours and 50 minutes. If he rows 2 miles an hour and walks 3 miles an hour, at what point should he land?

45. How long must a rectangular field be in order that it shall contain 3 acres and have a perimeter of 100 rods?

46. What must be the dimensions of a rectangular field in order that it shall contain 10 acres and have a perimeter of 160 rods?

47. Solve the system of equations

$$\begin{aligned} x^2 + y^2 &= 12, \\ xy &= 2. \end{aligned}$$

48. Solve the system of equations

$$\begin{aligned} x^2 + y^2 &= 18, \\ x + y &= 6. \end{aligned}$$

49. Solve the system of equations

$$\frac{x^2}{4}+\frac{y^2}{9}=1,$$

$$\frac{x^2}{4}-\frac{y^2}{9}=1.$$

50. Solve the system of equations

$$x^3-y^3=4,$$

$$x-y=1.$$

51. What must be the dimensions of a rectangular field in order that it shall contain 3 acres and that the greatest distance from any point in its boundary to any other point in its boundary be 34 rods?

APPENDIX

THE FACTOR THEOREM

1. Certain polynomials of degree greater than 2 that cannot be readily factored otherwise can be factored by the use of a theorem that is explained in the Second Course. This theorem says that a polynomial in x is divisible by $x - a$ if it equals zero when a is put in place of x. For example, we can see by trial that the polynomial $x^3 + 2x^2 - x - 2$ becomes equal to zero when 1 is put for x. We know therefore from the theorem that it is divisible by $x - 1$. Another factor can now be found by dividing the given polynomial by $x - 1$, and this division will incidentally verify the statement that $x - 1$ is a factor of $x^3 + 2x^2 - x - 2$.

In the application of this theorem to the factoring of polynomials, the number a must be determined by trial.

Example. Find the factors of $x^3 + 9x^2 + 26x + 24$.

We see by trial that this is equal to zero when -2 is put in place of x. We know therefore that $x - (-2)$, or $x + 2$, is a factor. Then we find by division that

$$x^3 + 9x^2 + 26x + 24 = (x + 2)(x^2 + 7x + 12).$$

But $$x^2 + 7x + 12 = (x + 3)(x + 4).$$

Hence $$x^3 + 9x^2 + 26x + 24 = (x + 2)(x + 3)(x + 4).$$

EXERCISES

Factor the following polynomials:

1. $x^3 - 6x^2 + 11x - 6$.
2. $x^3 - 7x + 6$.
3. $x^4 - 4x^3 - 15x^2 + 58x - 40$.
4. $6x^3 - 7x^2 - x + 2$.
5. $9x^3 + 6x^2 - 29x - 10$.
6. $4y^3 - 3y^2 - 12y - 5$.

7. x^3+2x^2+2x+1.
8. $4a^3+4a^2-23a-30$.
9. $4a^3+8a^2-27a-45$.
10. $x^3+x^2-10x+8$.
11. $a^4+2a^3-13a^2-38a-24$.
12. b^3-5b+4.
13. y^3-8y+7.
14. c^3+4c^2+c-6.
15. $3x^3-x^2-x-1$.
16. $6x^3-5x^2-21x-10$.
17. $9a^3+a+10$.
18. $9a^3-19a+10$.
19. $6x^3-5x^2-12x-4$.
20. $3m^3+8m^2+5m+2$.
21. $x^4-4x^3-5x^2+36x-36$.
22. a^4-25a^2+144.
23. $x^4+6x^3-3x^2-56x-48$.
24. $y^3+10y^2+29y+20$.
25. $x^4+10x^3+9x^2-160x-400$.
26. a^3-7a-6.
27. $a^3-a^2-17a-15$.
28. $m^3-5m^2-2m+24$.
29. $r^3+17r^2+47r-65$.
30. $x^4+10x^3+35x^2+50x+24$.

2. The Sum and Difference of Two Like Powers. — The factor theorem enables us to determine readily whether such expressions as x^n+a^n and x^n-a^n contain factors of the form $x+a$ or $x-a$.

Consider first x^n-a^n. This is equal to zero when a is put for x. It is therefore divisible by $x-a$, and we find by division that

$$x^n-a^n=(x-a)(x^{n-1}+x^{n-2}a+x^{n-3}a^2+\cdots+xa^{n-2}+a^{n-1}).$$

The succession of dots in this expression is to be read "and so on to."

If we put $-a$ for x, we see that x^n-a^n becomes equal to zero provided that n is an even integer. Hence in this case x^n-a^n is divisible by $x-(-a)=x+a$, and we find by division that

$$x^n-a^n=(x+a)(x^{n-1}-x^{n-2}a+x^{n-3}a^2-\cdots+xa^{n-2}-a^{n-1}).$$

The second factor here is like the second factor in the preceding case, with the exception that the signs of the terms are alternately positive and negative.

If n is an odd integer, $x^n - a^n$ is not zero when $-a$ is put for x. Hence $x^n - a^n$ is not divisible by $x + a$, since $x + a$ is zero when $-a$ is put for x, and zero times any number is zero.

If n is an even integer, the expression $x^n + a^n$ is not zero when either a or $-a$ is put for x. Hence in this case $x^n + a^n$ is not divisible by either $x + a$ or $x - a$.

When n is odd, $x^n + a^n$ is zero when $-a$ is put for x, and is not zero when a is put for x. In this case, therefore, $x^n + a^n$ is divisible by $x + a$, but not by $x - a$. We find by division that

$$x^n + a^n = (x + a)(x^{n-1} - x^{n-2}a + x^{n-3}a^2 - \cdots - xa^{n-2} + a^{n-1}).$$

3. Recapitulation. — I. *$x^n - a^n$ is divisible by $x - a$ whatever positive integral value n may have.*

II. *$x^n - a^n$ is divisible by $x + a$ when n is a positive even integer.*

III. *$x^n - a^n$ is not divisible by $x + a$ when n is a positive odd integer.*

IV. *$x^n + a^n$ is not divisible by either $x - a$ or $x + a$ when n is a positive even integer.*

V. *$x^n + a^n$ is divisible by $x + a$, but not by $x - a$, when n is a positive odd integer.*

The forms $x^2 - a^2$, $x^3 + a^3$, $x^3 - a^3$, with which the student is familiar, are special cases of these more general forms.

When n is a positive even integer, it is better to factor $x^n - a^n$ as the difference of two squares.

EXERCISES

Factor the following expressions:

1. $x^2 - a^2$.
2. $x^3 + a^3$.
3. $x^4 - a^4$.
4. $x^4 + a^4$.
5. $x^5 + y^5$.
6. $a^5 - b^5$.
7. $b^4 - 81$.
8. $a^6 - b^6$.
9. $y^5 - 1$.
10. $a^7 + b^7$.
11. $a^7 - b^7$.
12. $a^3 + 8x^6$.
13. $m^8 - n^8$.

14. x^5+1.

15. $1-b^4$.

16. a^6-64.

17. $243\,a^5-32$.

18. $32\,x^5+243\,y^5$.

19. $x^6-729\,y^{12}$.

20. $a^{4x}-b^{4x}$.

21. $x^{5n}-32\,y^{5n}$.

22. $x^{5a}+243$.

23. $(a+b)^4-16$.

24. $(a+b)^5+(x+y)^5$.

25. $(m+n)^5-32(x-y)^5$.

THE BINOMIAL THEOREM

4. The student can verify the following identities:

$$(a+b)^2=a^2+2\,ab+b^2.$$
$$(a+b)^3=a^3+3\,a^2b+3\,ab^2+b^3.$$
$$(a+b)^4=a^4+4\,a^3b+6\,a^2b^2+4\,ab^3+b^4.$$
$$(a+b)^5=a^5+5\,a^4b+10\,a^3b^2+10\,a^2b^3+5\,ab^4+b^5.$$

Indeed, he is already familiar with the first two.

Now these identities are special cases of a general law which can be stated as follows and is known as the **Binomial Theorem,** since it has to do with the powers of a binomial.

$$(a+b)^n=a^n+na^{n-1}b+\frac{n(n-1)}{1\cdot 2}a^{n-2}b^2+\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}a^{n-3}b^3+\cdots$$
$$+\frac{n(n-1)\cdots(n-r+1)}{1\cdot 2\cdot 3\cdots r}a^{n-r}b^r+\cdots+b^n.\quad\text{(I)}$$

Here n stands for any positive integer.

A proof of this theorem will be given in the College Algebra.

The right member of this identity is called the expansion of $(a+b)^n$.

The binomial theorem, as given in Formula I, may be stated as follows:

The expansion of $(a+b)^n$ *contains* $(n+1)$ *terms.*

The first term is a^n *and the last is* b^n.

The second term is $na^{n-1}b$.

The exponent of a in any term after the first one is one less than it is in the preceding term.

The exponent of b in any term after the second is one more than it is in the preceding term.

The coefficient of any term after the second is obtained from the preceding term by dividing the product of the coefficient of this term and the exponent of a by one more than the exponent of b.

It should be observed that the sum of the exponents of a and b in any term is n.

5. Definition. If r is any positive integer, the product of all the integers from 1 to r inclusive is called **factorial** r and is represented by the symbol $r\,!$ Sometimes the symbol $\underline{|r}$ is used. Thus

$$4\,! = 1 \cdot 2 \cdot 3 \cdot 4 = 24.$$
$$5\,! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120.$$

6. If r is any integer from 1 to n inclusive, the $(r+1)$th term in the expansion of $(a+b)^n$ is

$$\frac{n(n-1)(n-2)\cdots(n-r+1)}{r\,!}a^{n-r}b^r. \qquad \text{(II)}$$

Thus the fourth term in the expansion of $(a+b)^7$ is

$$\frac{7 \cdot 6 \cdot 5}{3\,!}a^{7-3}b^3 = 35\,a^4b^3.$$

Here $n = 7$ and $r + 1 = 4$. Hence $r = 3$.

By means of Formula II we can write down any term in the expansion of $(a+b)^n$ without writing the preceding terms.

7. The expansion of $(a-b)^n$ can be obtained from that of $(a+b)^n$ by putting $-b$ in place of b.

Example 1. Expand $(2x+3y)^7$.

Here $a = 2x$, $b = 3y$, $n = 7$.

$$\begin{aligned}(2x+3y)^7 &= (2x)^7 + 7(2x)^6 3y + \frac{7 \cdot 6}{2\,!}(2x)^5(3y)^2 + \frac{7 \cdot 6 \cdot 5}{3\,!}(2x)^4(3y)^3 \\ &\quad + \frac{7 \cdot 6 \cdot 5 \cdot 4}{4\,!}(2x)^3(3y)^4 + \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3}{5\,!}(2x)^2(3y)^5 \\ &\quad + \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2}{6\,!}2x(3y)^6 + (3y)^7 \\ &= 128\,x^7 + 1344\,x^6y + 6048\,x^5y^2 + 15120\,x^4y^3 + 22680\,x^3y^4 \\ &\quad + 20412\,x^2y^5 + 10206\,xy^6 + 2187\,y^7.\end{aligned}$$

Example 2. Expand $(3a-4b)^5$.

Here $a = 3a$, $b = -4b$, $n = 5$.

$$(3a-4b)^5 = (3a)^5 + 5(3a)^4(-4b) + \frac{5\cdot 4}{2!}(3a)^3(-4b)^2$$
$$+\frac{5\cdot 4\cdot 3}{3!}(3a)^2(-4b)^3 + \frac{5\cdot 4\cdot 3\cdot 2}{4!}(3a)(-4b)^4 + (-4b)^5$$
$$= 243a^5 - 1620a^4b + 4320a^3b^2 - 5760a^2b^3 + 3840ab^4 - 1024b^5.$$

EXERCISES

Expand the following by the Binomial Theorem:

1. $(x+y)^8$.
2. $(x-y)^8$.
3. $(a-2)^4$.
4. $(2-a)^4$.
5. $(3-x)^5$.
6. $(b+1)^7$.
7. $(a+x)^6$.
8. $(2a+4c)^5$.
9. $(2a^2-b^2)^6$.
10. $(3a^2b+2c^3)^5$.
11. $(1-x)^9$.
12. $(x-1)^9$.
13. $(1+x)^{10}$.
14. $(x+1)^{10}$.
15. $(m^3-n^3)^5$.
16. $\left(\frac{a^3}{b}+\frac{3b^2}{a}\right)^4$.
17. $\left(\frac{x^2}{y}+\frac{y^2}{x}\right)^4$.
18. $(3a^2b^4-\frac{1}{2})^4$.
19. $\left(x^3+\frac{1}{x^5}\right)^4$.
20. $(3x-5y)^5$.
21. $(3+2a)^6$.
22. $(xy+yz)^4$.
23. $(x+\frac{1}{2}y)^5$.
24. $(\frac{1}{4}x-\frac{1}{8}y)^4$.

In each of the following exercises write the term asked for without finding any other term:

25. The 3rd term of $(1-x)^7$.
26. The 6th term of $(a+b)^9$.
27. The 6th term of $(a-b)^9$.
28. The 5th term of $(3a-2b)^8$.
29. The 8th term of $\left(2x+\frac{1}{2x}\right)^{11}$.
30. The 4th term of $(4x^2y-5xy^2)^{10}$.

FRACTIONAL AND NEGATIVE EXPONENTS

8. The fundamental laws in the use of positive integral exponents are expressed by the formulæ:

I. $a^m \cdot a^n = a^{m+n}$.

II. $a^m \div a^n = a^{m-n}$. (When n is less than m.)

III. $(a^m)^n = a^{mn}$.

IV. $(ab)^m = a^m b^m$.

V. $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$.

Now heretofore we have considered only positive integral exponents, since the purpose of an exponent was to indicate how many times a number was to be taken as a factor. It turns out, however, to be possible to use fractional and negative exponents in a simple and natural way that is also consistent with these laws.

9. If we agree that $a^{\frac{1}{2}}$ shall be a symbol for a number such that

$$a^{\frac{1}{2}} \cdot a^{\frac{1}{2}} = a,$$

we shall have a definition that is consistent with the first of these laws.

But this is equivalent to saying that $a^{\frac{1}{2}}$ is one of two equal factors of a. We accordingly agree that this symbol shall represent the same number as $\sqrt{a}$.

More generally we agree that the symbol $a^{\frac{p}{q}}$ shall stand for the same number as $\sqrt[q]{a^p}$. This is equivalent to saying that $a^{\frac{p}{q}}$ is a qth root of a^p. It is also equivalent to the pth power of a qth root of a. That is,

$$a^{\frac{p}{q}} = \sqrt[q]{a^p} = (\sqrt[q]{a})^p.$$

NOTE. — The symbol $a^{\frac{p}{q}}$ is read "a to the power p divided by q," or "a exponent p divided by q."

When a fractional exponent is used in connection with a number, we shall accordingly look upon the denominator of this exponent as indicating a root and upon the numerator as indicating a power.

For example,

$$9^{\frac{1}{2}}=\sqrt{9}=3\,;\ 2^{\frac{3}{4}}=\sqrt[4]{2^3}=\sqrt[4]{8}=(\sqrt[4]{2})^3\,;\ (a-b)^{\frac{4}{3}}=\sqrt[3]{(a-b)^4}=(\sqrt[3]{a-b})^4.$$

10. It would have been in accordance with the fundamental laws of exponents to have agreed that $9^{\frac{1}{2}} = -3$. But it is convenient to call the positive even roots of positive numbers and the odd roots of negative numbers **principal roots**, and to use radical signs and fractional exponents to indicate these principal roots. Thus we say that $9^{\frac{1}{2}} = \sqrt{9} = 3$, and $-9^{\frac{1}{2}} = -\sqrt{9} = -3$.

11. Now $\dfrac{a^m}{a^m} = 1$,

and also Fundamental Law II suggests that we indicate this quotient by a^{m-m} or a^0. It is natural therefore to agree to use the symbol a^0 to stand for the number 1, regardless of the value of a.

$$a^0 = 1.$$

Fundamental Law I suggests that we use the symbol a^{-m} to stand for a number such that

$$a^{-m} \cdot a^m = a^{-m+m} = a^0 = 1,$$

or

$$a^{-m} = \frac{1}{a^m}.$$

We agree to follow this suggestion, and accordingly a number with a negative exponent — or, as we shall say, — raised to a negative power, is equal to the reciprocal of this number raised to the corresponding positive power.

The value of a fraction will not be affected if any factor is transferred from the numerator to the denominator, or from the denominator to the numerator, provided the sign of its exponent is changed. Thus:

$$\frac{x^2}{y^3z^4}=\frac{x^2y^{-3}}{z^4}=\frac{1}{x^{-2}y^3z^4};$$

$$\frac{(x+2\,y)^2\,(x-3\,y)^5}{(a-b)^3\,(a+4\,b)^2}=\frac{(x+2\,y)^2\,(x-3\,y)^5\,(a-b)^{-3}}{(a+4\,b)^2}$$

$$=(x+2\,y)^2\,(x-3\,y)^5\,(a-b)^{-3}\,(a+4\,b)^{-2}.$$

The student is cautioned that this applies only to factors of the numerator and of the denominator, and not to separate terms. Thus:

$$\frac{x^3+y^3}{z^2}\text{ is not equal to }\frac{x^3}{y^{-3}z^2}\text{ or to }\frac{x^3}{z^2+y^{-3}}.$$

EXERCISES

Write each of the following expressions without the use of the radical sign:

1. $\sqrt{19}$.
2. $\sqrt[5]{24}$.
3. $\sqrt[3]{3}$.
4. $6\sqrt{(x+y)^3}$.
5. $\frac{1}{5}\sqrt[5]{x^3y^5z^{-2}}$.
6. $\sqrt[3]{\frac{1}{8}}$.
7. $\sqrt[4]{5^{-3}}$.
8. $\sqrt{21}$.
9. $\sqrt[5]{a^{10}}$.
10. $\sqrt[6]{x^5y^4}$.
11. $\sqrt{(x-3\,y)^3(3\,x+4\,y)}$.
12. $\sqrt{a^4+6\,a^3b-5\,a^2b^2-2\,ab^3+7\,b^4}$.
13. $\frac{1}{3\,b}\sqrt[3]{\frac{a^4}{b^2}}$.
14. $\sqrt[4]{\frac{(4\,a+3\,b)^3(2\,a-b)^4}{(x+y)^5(3\,x-2\,y)^6}}$.
15. $\sqrt{\frac{19}{20}}$.
16. $\sqrt[5]{(x-y)^{-3}}$.
17. $\sqrt[7]{(a+b)^2(a-b)^3}$.
18. $\sqrt{a+b+x}$.
19. $\sqrt[5]{2^3\cdot 4^7}$.
20. $\sqrt[4]{9(x+y)^2}$.

Write each of the following expressions without the use of fractional or negative exponents:

21. $\frac{3}{2^{-\frac{1}{2}}}$.
22. $x^{\frac{2}{3}}y^{-\frac{1}{3}}$.
23. $3^{\frac{1}{2}}x^{\frac{4}{5}}$.
24. $8^{-\frac{1}{3}}$.
25. $25^{-\frac{1}{2}}$.

26. $(-\frac{4}{5})^{-\frac{4}{5}}$.

27. $5^{\frac{1}{4}}x^{\frac{2}{3}}y^{\frac{7}{8}}z^{-\frac{2}{7}}$.

28. $3^{\frac{1}{6}}\cdot 4^{\frac{1}{2}}$.

29. $\left(\frac{a}{b}\right)^{-3}$.

30. $3^{-2}x^2y^{-2}$.

31. $(\frac{2}{11})^{\frac{1}{3}}$.

32. $(\frac{11}{2})^{-\frac{1}{3}}$.

33. $(x^3+2\,x^2y-5\,xy^2+3\,y^3)^{\frac{1}{3}}$.

34. $\left(\frac{1}{a^2+4\,ab+8\,b^2}\right)^{-4}$.

35. $(2\,x-3\,y)^{\frac{1}{4}}(3\,x-2\,y)^{\frac{2}{3}}(2\,x+3\,y)^{-\frac{1}{2}}$.

36. $\frac{x^3+y^{-3}}{2}$.

37. $\frac{a^x+a^{-x}}{2}$.

38. $\frac{2}{a^x-a^{-x}}$.

39. $\frac{a^x+a^{-x}}{a^x-a^{-x}}$.

40. $\frac{m+m^{-1}}{m-m^{-1}}$.

Write each of the following expressions without negative exponents and in as simple a form as possible:

41. $16^{\frac{1}{2}}$.

42. $16^{-\frac{1}{2}}$.

43. 16^2.

44. 16^{-2}.

45. 3^{-3}.

46. $8^{-\frac{1}{3}}$.

47. $25^{\frac{1}{2}}$.

48. $\frac{1}{125^{-\frac{1}{3}}}$.

49. $(\frac{36}{49})^{\frac{1}{2}}$.

50. $(.64)^{-\frac{1}{2}}$.

51. 8456^0.

52. $\frac{4}{67^0}$.

53. $8^{\frac{2}{3}}$.

54. $16^{-\frac{3}{2}}$.

55. $(\frac{1}{5})^0$.

56. $(-\frac{1}{8})^{\frac{1}{3}}$.

57. $(-\frac{1}{8})^{-\frac{1}{3}}$.

58. $2^0\cdot 3^2\cdot 5^{-2}$.

59. $3\,a^{-2}$.

60. $\frac{2-2^{-2}}{2+2^{-2}}$.

61. $\frac{1}{x^{-3}+y^{-3}}$.

62. $7\,x^{-4}$.

63. $(7\,x)^{-4}$.

64. $(64\,a^{-9}b^6c^{12})^{-\frac{2}{3}}$.

Perform the following multiplications and divisions in accordance with the usual rules:

Multiply:

65. $x^{\frac{1}{2}}+y^{\frac{1}{2}}$ by $x^{\frac{1}{2}}-y^{\frac{1}{2}}$.

66. $x+x^{\frac{1}{2}}y^{\frac{1}{2}}+y$ by $x^{\frac{1}{2}}-y^{\frac{1}{2}}$.

67. $a^{\frac{2}{3}}+4\,a^{\frac{1}{3}}+2$ by $3\,a^{\frac{2}{3}}-a^{\frac{1}{3}}-1$.

68. $a^{\frac{1}{2}}+a^{\frac{1}{4}}b^{\frac{1}{2}}+b$ by $a^{\frac{1}{4}}-b^{\frac{1}{2}}$.

69. $x^{\frac{2}{3}} - x^{\frac{1}{3}}y^{\frac{1}{3}} + y^{\frac{2}{3}}$ by $x^{\frac{2}{3}} + x^{\frac{1}{3}}y^{\frac{1}{3}} + y^{\frac{2}{3}}$.

70. $\sqrt{x^5} + \sqrt{y^5}$ by $x^{\frac{5}{2}} - y^{\frac{5}{2}}$.

71. $m^{\frac{8}{3}} + m^{\frac{4}{3}}n^{\frac{4}{3}} + n^{\frac{8}{3}}$ by $m^{\frac{4}{3}} - n^{\frac{4}{3}}$.

72. $a^{\frac{4}{5}} + a^{\frac{3}{5}}b^{\frac{1}{5}} + a^{\frac{2}{5}}b^{\frac{2}{5}} + a^{\frac{1}{5}}b^{\frac{3}{5}} + b^{\frac{4}{5}}$ by $a^{\frac{1}{5}} - b^{\frac{1}{5}}$.

73. $x^{\frac{3}{2}} - xy^{\frac{1}{2}} + x^{\frac{1}{2}}y - y^{\frac{3}{2}}$ by $x^{\frac{1}{2}} + y^{\frac{1}{2}}$.

74. $m^{\frac{4}{3}} - m^{\frac{2}{3}}n^{\frac{2}{3}} + n^{\frac{4}{3}}$ by $m^{\frac{2}{3}} + n^{\frac{2}{3}}$.

Divide:

75. $a - b$ by $a^{\frac{1}{2}} + b^{\frac{1}{2}}$.

76. $x^3 - y^3$ by $x^{\frac{3}{2}} - y^{\frac{3}{2}}$.

77. $a^{\frac{3}{4}} - b^{\frac{3}{2}}$ by $a^{\frac{1}{4}} - b^{\frac{1}{2}}$.

78. $m^2 + n^5$ by $m^{\frac{2}{5}} + n^{\frac{5}{5}}$.

79. $b - 3x$ by $b^{\frac{1}{2}} - 3^{\frac{1}{2}}x^{\frac{1}{2}}$.

80. $15a^{\frac{1}{2}} - a^{\frac{1}{4}}b^{\frac{1}{2}} - 6b$ by $3\sqrt[4]{a} - 2\sqrt{b}$.

81. $2x^{\frac{4}{3}} - 5x^{\frac{2}{3}}y^{\frac{2}{3}} - 3y^{\frac{4}{3}}$ by $2x^{\frac{2}{3}} + y^{\frac{2}{3}}$.

82. $x + y$ by $x^{\frac{1}{5}} + y^{\frac{1}{5}}$.

83. $8a^3 - 5b^3$ by $8^{\frac{1}{4}}a^{\frac{3}{4}} - 5^{\frac{1}{4}}b^{\frac{3}{4}}$.

84. $16m^2 - 81n^2$ by $2m^{\frac{1}{2}} - 3n^{\frac{1}{2}}$.

RADICALS

12. Definitions. — In a radical of the form $b\sqrt[n]{a}$, a is called the **radicand**, and b the **coefficient**. (See § 137.)

A radical is said to be in its simplest form when (*a*) the radicand is an integer, or a polynomial with integral coefficients; (*b*) the radicand contains no factors raised to powers equal to, or greater than, the index of the radical; (*c*) the radicand is not a power whose index has a factor in common with the index of the radical.

Thus $\sqrt{\frac{2}{3}}$, $\sqrt[3]{(a-b)^4}$, and $\sqrt[6]{a^4}$ are not in their simplest forms.

13. Radicals can sometimes be put into their simplest forms by the application of certain of the following reductions:

(1) **Reduction of a Fractional Radicand to the Integral Form.** This is illustrated by the following examples:

$$\sqrt{\frac{2}{3}} = \sqrt{\frac{6}{9}} = \frac{\sqrt{6}}{\sqrt{9}} = \frac{\sqrt{6}}{3}.$$

$$\sqrt[4]{\frac{a}{b}} = \sqrt[4]{\frac{ab^3}{b^4}} = \frac{\sqrt[4]{ab^3}}{b}.$$

(2) **Removal of Factors from the Radicand.**

Examples.

$$\sqrt{8} = \sqrt{4 \cdot 2} = \sqrt{4} \cdot \sqrt{2} = 2\sqrt{2}.$$

$$\sqrt[3]{108} = \sqrt[3]{27 \cdot 4} = \sqrt[3]{27} \cdot \sqrt[3]{4} = 3\sqrt[3]{4}.$$

$$\sqrt[3]{(a-b)^4} = \sqrt[3]{(a-b)^3(a-b)} = \sqrt[3]{(a-b)^3} \cdot \sqrt[3]{a-b} = (a-b)\sqrt[3]{a-b}.$$

(3) **Lowering of the Index of the Radical.**

Examples.

$$\sqrt[6]{a^4} = a^{\frac{4}{6}} = a^{\frac{2}{3}} = \sqrt[3]{a^2}.$$

$$\sqrt[4]{a^2 - 2ab + b^2} = (a-b)^{\frac{2}{4}} = (a-b)^{\frac{1}{2}} = \sqrt{a-b}.$$

In certain problems it is desirable to introduce factors into the radicand or to increase the index of the radical. These reductions are the inverses of (2) and (3).

Examples. $4\sqrt[3]{2} = \sqrt[3]{64} \cdot \sqrt[3]{2} = \sqrt[3]{128}.$ $\sqrt{5} = 5^{\frac{1}{2}} = 5^{\frac{3}{6}} = \sqrt[6]{5^3} = \sqrt[6]{125}.$

EXERCISES

Reduce each of the following radicals to its simplest form:

1. $\sqrt{36}$.
2. $\sqrt{18}$.
3. $\sqrt{8}$.
4. $\sqrt{27}$.
5. $\sqrt{250}$.
6. $\sqrt{45}$.
7. $\sqrt[4]{64}$.
8. $\sqrt[3]{54}$.
9. $\sqrt[4]{2^5 \cdot 3^2}$.
10. $\sqrt[6]{27 \cdot 64}$.
11. $\sqrt[3]{135}$.
12. $\sqrt{\frac{1}{2}}$.

13. $\sqrt{\frac{1}{3}}$.

14. $\sqrt{\frac{5}{6}}$.

15. $\sqrt[3]{\frac{1}{2}}$.

16. $\sqrt[3]{\frac{2}{3}}$.

17. $\sqrt[3]{\frac{4}{5}}$.

18. $a^{\frac{3}{4}}b^{\frac{5}{4}}$.

19. $3\sqrt{(x-y)^5}$.

20. $\sqrt{a^3+3a^2b+3ab^2+b^3}$.

21. $\sqrt[3]{8(a+b)^6}$.

22. $\sqrt{\frac{a}{b}}$.

23. $\sqrt[3]{16(x-y)^4}$.

24. $\sqrt[3]{\frac{a}{b}}$.

25. $\sqrt[4]{4(x^2-6xy+9y^2)}$.

26. $a^{\frac{2}{3}}m^{\frac{5}{3}}n^{\frac{7}{3}}$.

27. $\sqrt[5]{64a^6b^{12}}$.

28. $2^{\frac{1}{2}}x^{\frac{3}{2}}y^{\frac{5}{2}}$.

29. $\sqrt[6]{(2a+3b)^3}$.

30. $\sqrt[3]{\frac{x^2yz}{a^4b^4}}$.

14. Addition and Subtraction of Radicals. — Two or more radicals are said to be similar if, when simplified, they have the same index and the same radicand.

For example, $2\sqrt[3]{5}$ and $3\sqrt[3]{5}$ are similar as are also $\sqrt{a^3b}$ and $3\sqrt{ab}$.

An expression involving two or more radicals can sometimes be simplified by first simplifying each radical and then combining the similar radicals in the way illustrated in the following examples:

$$2\sqrt{98}-3\sqrt{50}+\sqrt{72}=14\sqrt{2}-15\sqrt{2}+6\sqrt{2}=5\sqrt{2}.$$

$$\begin{aligned}2\sqrt{98}-50\sqrt{3}+\sqrt{32}-\sqrt{108}&=14\sqrt{2}-50\sqrt{3}+4\sqrt{2}-6\sqrt{3}\\&=18\sqrt{2}-56\sqrt{3}.\end{aligned}$$

It should be observed that the sum or difference of two dissimilar radicals cannot be expressed as a single radical.

15. Multiplication of Radicals. — The product of two radicals with a common index is a radical with the same index whose coefficient and radicand are equal respectively to the products of the coefficients and of the radicands of the factors.

Thus, $$a\sqrt[n]{b}\cdot c\sqrt[n]{d}=ac\sqrt[n]{bd}.$$

If the radicals do not have the same index, they should be reduced to equal radicals with a common index, in accordance with the following illustration.

$$\sqrt[3]{5} \cdot \sqrt{6} = 5^{\frac{1}{3}} \cdot 6^{\frac{1}{2}} = 5^{\frac{2}{6}} \cdot 6^{\frac{3}{6}} = \sqrt[6]{25} \cdot \sqrt[6]{216} = \sqrt[6]{5400};$$

$$\sqrt{3} \cdot \sqrt[3]{4} = \sqrt[6]{27} \cdot \sqrt[6]{16} = \sqrt[6]{432}.$$

The product of such expressions as $\sqrt{a} + \sqrt{b} - \sqrt{c}$ and $\sqrt{a} - \sqrt{b} + \sqrt{c}$ can be formed by applying the usual rules for the multiplication of polynomials in connection with the principles just stated.

Thus

$$\begin{array}{rlllll}
 & \sqrt{a} & +\sqrt{b} - \sqrt{c} & & & \\
 & \sqrt{a} & -\sqrt{b} + \sqrt{c} & & & \\
\hline
 & a & +\sqrt{ab} & -\sqrt{ac} & & \\
 & & -\sqrt{ab} & & -b & +\sqrt{bc} \\
 & & & +\sqrt{ac} & & +\sqrt{bc} - c \\
\hline
\textit{Product} & a & & & -b & +2\sqrt{bc} - c
\end{array}$$

16. Division of Radicals. — The quotient of two radicals with a common index is a radical with this same index whose coefficient and radicand are equal respectively to the quotients of the coefficients and of the radicands of the dividend and the divisor.

Thus $$a\sqrt[n]{b} \div c\sqrt[n]{d} = \frac{a}{c}\sqrt[n]{\frac{b}{d}}.$$

If the radicals do not have the same index, they should first be reduced to equal radicals with a common index.

17. Rationalizing the Denominator of a Fraction. — Certain fractions whose denominators are irrational can be changed into equal fractions with rational denominators.

For example,

$$\frac{3}{\sqrt{2}} = \frac{3\sqrt{2}}{\sqrt{2} \cdot \sqrt{2}} = \frac{3\sqrt{2}}{2}.$$

$$\frac{a}{\sqrt{a} + \sqrt{b}} = \frac{a(\sqrt{a} - \sqrt{b})}{(\sqrt{a} + \sqrt{b})(\sqrt{a} - \sqrt{b})} = \frac{a(\sqrt{a} - \sqrt{b})}{a - b}.$$

Such a change can always be made when the denominator is a single radical or the sum of two terms one of which is a square root and the other either a square root or rational.

It is in general best to simplify all the radicals in a problem before attempting to perform any operations with them.

EXERCISES

Simplify the following expressions as much as possible:

1. $5\sqrt{18}-2\sqrt{8}+6\sqrt{72}$.
2. $\sqrt{75}+6\sqrt{108}+5\sqrt{12}$.
3. $10\sqrt{150}-7\sqrt{96}-3\sqrt{54}$.
4. $\sqrt[3]{40}+\sqrt[3]{135}+\sqrt[3]{320}+\sqrt[3]{625}$.
5. $3\sqrt{80}-2\sqrt{75}+\sqrt{108}+5\sqrt{20}$.
6. $3\sqrt{\frac{1}{2}}-\sqrt{50}+4\sqrt{162}$.
7. $3\sqrt{a^3}-2\sqrt{ab^2}+4\sqrt{a^5c^2}$.
8. $3x^{\frac{5}{3}}-2x^{\frac{2}{3}}$.
9. $3\sqrt{\frac{12}{5}}-5\sqrt{\frac{4}{15}}-2\sqrt{\frac{5}{8}}$.
10. $\sqrt{(a^2-b^2)(a^2+2ab+b^2)}+b\sqrt[3]{a-b}$.
11. $3\sqrt{45}\cdot 4\sqrt{72}$.
12. $6ax^{\frac{2}{3}}\cdot 5ax^{-\frac{1}{2}}$.
13. $\sqrt{\frac{3}{64}}\cdot\sqrt{\frac{2}{7}}$.
14. $(\sqrt{a-b})^3(\sqrt{a+b})^3$.
15. $(\sqrt{a}-\sqrt{a-b})^2$.
16. $(10-\sqrt{5})(5+\sqrt{10})$.
17. $(\sqrt{2}+\sqrt{3}+\sqrt{5})(\sqrt{2}-\sqrt{3}-\sqrt{5})$.
18. $(\sqrt[3]{2}+\sqrt[3]{a})(\sqrt[3]{4}-\sqrt[3]{2a}+\sqrt[3]{a^2})$.

19. $\sqrt{35} \div \sqrt{\frac{5}{7}}$.

20. $6 \div \sqrt{2}$.

21. $5 \div \sqrt[3]{4}$.

22. $2\sqrt[n]{a} \div \sqrt[n]{2^n a}$.

23. $\sqrt[3]{a^2bc} \div \sqrt[3]{a^2b^2c}$.

24. $(\sqrt{8} + 5\sqrt{12} - 3\sqrt{14}) \div 2$.

25. $(\sqrt{2} + \sqrt{3} - \sqrt{5}) \div \sqrt{6}$.

26. $(3\sqrt[3]{5})^3$.

27. $(2\sqrt[4]{2})^5$.

28. $(\sqrt{2} - \sqrt{5})^2$.

29. $(2\sqrt[3]{4} + 5\sqrt[3]{6})^3$.

30. $\dfrac{16}{\sqrt{12}}$.

31. $\dfrac{4}{\sqrt[3]{a}}$.

32. $\dfrac{a+b}{\sqrt{x}}$.

33. $\dfrac{4}{\sqrt{5}-2}$.

34. $\dfrac{1}{\sqrt{5}+2}$.

35. $\dfrac{\sqrt{5}-\sqrt{2}}{\sqrt{5}+\sqrt{2}}$.

36. $\dfrac{1+\sqrt{6}}{1-\sqrt{6}}$.

37. $\dfrac{\sqrt{a}-\sqrt{b}}{\sqrt{a}+\sqrt{b}}$.

38. Is $1 - \sqrt{3}$ a root of the equation $x^3 - x - 1 = 0$?

39. Is $2 + 3\sqrt{5}$ a root of the equation $2x^3 + 3x^2 - 2x + 3 = 0$?

40. Is $\dfrac{-3+\sqrt{5}}{2}$ a root of the equation $x^2 + 3x + 1 = 0$?

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